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THE EFFECTS OF ERRORS OR VARIATIONS IN THE ARBITRARY CONSTANTS OF SIMULTANEOUS EQUATIONS

BY

GEORGE H. DELL

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THE EFFECTS OF ERRORS OR VARIATIONS IN THE ARBITRARY CONSTANTS OF SIMULTANEOUS EQUATIONS

I. Introduction

1. *Introductory.*—Many problems in science and engineering involve the use of quantities determined by means of simultaneous equations whose coefficients are subject to errors or variations. Examples in the field of structural engineering occur in the analysis of continuous frames, where the coefficients of the governing equations depend upon various ideal dimensions and elastic properties which are not exactly realized in the actual structure. A similar situation exists in connection with the determination or the interpretation of experimental data, if equations which are subject to observational errors or physical variations are involved. It is appropriate, therefore, that an inquiry be made into the effects of such errors.

The basis of the problem consists in the presence of errors of unknown signs and magnitudes in the arbitrary constants entering into the equations. These arbitrary constants are generally of a geometrical or physical nature, and include parametric, dimensional, and empirical constants.

In studying the effects of such errors upon the computed values of the unknowns, or upon given functions of the unknowns, the aim of the investigation may be either to determine the probable errors, or to calculate the maximum effects which may be produced by errors of assigned magnitudes in the arbitrary constants. In the former case, the probable errors of the arbitrary constants may be estimated or obtained by experiment; in the latter case, the magnitudes to be assigned to the errors in the arbitrary constants may be determined with reference to the corresponding probable errors, or may be assumed arbitrarily.

2. *Acknowledgments.*—The study reported herein was made under the auspices of the Department of Civil Engineering, of which Prof. W. C. Huntington is the head, and of the College of Engineering and the Engineering Experiment Station, of which M. L. Enger is Dean and Director. Credit for suggesting and encouraging the investigation is due to Hardy Cross, formerly Professor of Structural Engineering at the University of Illinois. Credit is also due to Nathan M. Newmark, Research Assistant Professor of Civil Engi-

neering, for a careful examination of the preliminary manuscript and for valuable advice regarding the content and arrangement of the material.

3. *Previous Contributions.*—Professor Blumenthal, at Aachen,* in investigating the precision of the solution of linear equations, derived an expression by means of which he was able to determine an upper bound to the error in the unknown which was most seriously affected by errors of restricted magnitudes in the coefficients. Professor A. Hertwig, also at Aachen,† derived an expression for the primary error in a given unknown of a system of linear equations in which the errors in the coefficients were taken at a fixed percentage, and showed how to determine the signs in order to obtain the maximum primary error. He also stated the fundamental formula defining the mean error of a given unknown (the mean error being a quantity related to the probable error) and derived an approximate method for evaluating the same. In addition, he investigated the "error-sensitiveness" of a given system of equations, applied to the analysis of statically indeterminate structures, with a view to determining the relative advantages of choosing different quantities as the redundants in the solution. For this purpose he used as a criterion the error in the determinant of the coefficients of the governing equations.

4. *Extent of the Investigation.*—The present treatment is believed to include several new features, among them being the derivation of general expressions for the primary error by the method of total differentials, the calculation of errors of higher orders, the distinction between independent and interdependent coefficients, and a discussion of the true maximum error. The investigation is also extended to non-linear equations.

The study deals specifically with systems containing n linearly independent equations in n unknowns, as follows:

(a) Linear equations with linear arbitrary constants.

(b) Non-linear equations with non-linear arbitrary constants.

No difficulty should be experienced in extending the treatment to linear equations with non-linear arbitrary constants or to other combinations.

The investigation is particularly concerned with the calculation of maximum errors, but since, in many applications, the probable

*Blumenthal, Ueber die Genauigkeit der Wurzeln linearer Gleichungen, Zeitsch. f. Math. u. Phys., Bd. 62, Heft 4.

†A. Hertwig, Die Fehlerwirkungen beim Aufloesen linearer Gleichungen und die Berechnung statisch unbestimmte Gebilde, Der Eisenbau, Vol. 8, 1917, pp. 110-132.

errors may be of primary interest, and since a considerable portion of the procedure is common to both quests, the steps involved in the determination of the probable errors are also briefly outlined and illustrated.

II. Basic Concepts

5. *Definitions.*—For purposes of definition, consider the following system of linear equations:

$$f_1 = a_{11}x_1 + \ldots + a_{1m}x_m + \ldots + a_{1n}x_n - \mu_1 = 0 \qquad (1)$$

. . . . . .

$$f_m = a_{m1}x_1 + \ldots + a_{mm}x_m + \ldots + a_{mn}x_n - \mu_m = 0 \qquad (1m)$$

. . . . . .

$$f_n = a_{n1}x_1 + \ldots + a_{nm}x_m + \ldots + a_{nn}x_n - \mu_n = 0 \qquad (1n)$$

where the coefficients $a_{11} \ldots a_{nn}$ contain the arbitrary constants $k_1 \ldots k_p$, which are subject to the errors or variations represented by the increments $\Delta k_1 \ldots \Delta k_p$, respectively; and $\mu_1 \ldots \mu_n$ are constants usually appearing on the right-hand side of the equations, and which hence will be referred to as the "right-hand terms." For the present, these quantities will be regarded as absolute constants.

Independent Coefficients. If none of the arbitrary constants $k_1 \ldots k_p$ appears in more than one of the aggregate number of the coefficients in the given system, the coefficients are defined as being independent.

Interdependent Coefficients. If any or all of the arbitrary constants $k_1 \ldots k_p$ appears in two or more of the aggregate number of the coefficients in the given system, the coefficients in question are defined as being interdependent.

Initial Values. The initial value of a given quantity is that value determined by using the nominal or mean values of the arbitrary constants. The initial values of the arbitrary constants, of the coefficients, and of the unknowns are denoted by $k_1^\circ \ldots k_p^\circ$, $a_{11}^\circ \ldots a_{nn}^\circ$, and $x_1^\circ \ldots x_n^\circ$, respectively.

Modified Values. The modified value of a given quantity is that value determined by assigning definite signs and magnitudes to the errors in the arbitrary constants. The modified values of the arbitrary constants, of the coefficients, and of the unknowns are denoted by $k_1' \ldots k_p'$, $a_{11}' \ldots a_{nn}'$, and $x_1' \ldots x_n'$, respectively.

Original Equations. The original equations are those in which the

coefficients have their initial values, $a^\circ_{11} \ldots . a^\circ_{nn}$; their solution determines the initial values of the unknowns, $x^\circ_1 \ldots . x^\circ_n$.

Revised Equations. The revised equations are those in which the coefficients have their modified values, $a'_{11} \ldots . a'_{nn}$; their solution determines the modified values of the unknowns, $x'_1 \ldots . x'_n$.

True Errors. The true error in a given quantity is defined as the difference between the modified and the initial values of that quantity.

The true error in a given coefficient, a°_{ij}, is

$$\Delta a^\circ_{ij} = a'_{ij} - a^\circ_{ij};$$

similarly, the true error in a given unknown, x°_m, is

$$\Delta x^\circ_m = x'_m - x^\circ_m.$$

The true error may generally be represented by a Taylor's series.

Primary Errors. The primary error in a given quantity is the total first-order differential of that quantity. It is equivalent to the first term of the Taylor's series which expresses the true error. The symbol δx_m is used to denote the primary error in x°_m.

The primary error is expressible as a linear function of the errors $\Delta k_1 \ldots . \Delta k_p$ with the aid of the initial values, $k^\circ_1 \ldots . k^\circ_p$, $a^\circ_{11} \ldots . a^\circ_{nn}$, and $x^\circ_1 \ldots . x^\circ_n$.

Maximum Primary Error. The maximum primary error in a given quantity is the numerical value of the primary error resulting from assigning to the errors in the arbitrary constants such signs as will make the separate terms in the expression for the primary error cumulative in their effects.

(Note: The term "maximum" is used here and in subsequent definitions to denote "greatest," though not necessarily in the sense of an extremum, that is, where the derivatives vanish.)

Errors of Higher Orders. Following the primary error, the successive terms in the Taylor's series for the true error in a given quantity represent the errors of higher orders. The symbols $\delta^2 x_m$, $\delta^3 x'_m$, etc., are used to denote the errors of higher orders in x°_m.

Apparent Maximum Error. The apparent maximum error in a given quantity is the true error obtained by using with the errors in the arbitrary constants a particular combination of signs which is presumed to yield the maximum error.

True Maximum Error. The true maximum error in a given quantity is the true error obtained by using that particular combination of signs with the errors (of limited magnitudes) in the arbitrary constants which will yield the greatest possible error in that quantity.

Error-coefficients. The coefficients of the increments $\Delta k_1 \ldots . \Delta k_p$ in the expression for the primary error of a given quantity, stated in numerical or in literal terms, are called "error-coefficients." Since the primary error is identical with the total first-order differential, the error-coefficients are differential coefficients, or partial derivatives.

Probable Error. The probable error is a quantity pertaining to the probability curve or error function. It is defined as that error which is just as likely as not to be exceeded.*

6. *General Relations.*—If from a function $x(r, s, t, \ldots .)$ of several independent variables $r, s, t, \ldots .$, it is desired to develop the function

$$x^\circ + \Delta x^\circ = x(r^\circ + \Delta r, s^\circ + \Delta s, t^\circ + \Delta t, \ldots .),$$

where $x^\circ = x(r^\circ, s^\circ, t^\circ, \ldots .)$, the indicated expansion is given by the following Taylor's series:

$$x^\circ + \Delta x^\circ = x^\circ + \left(\frac{\partial x}{\partial r}\Delta r + \frac{\partial x}{\partial s}\Delta s + \frac{\partial x}{\partial t}\Delta t + \ldots .\right)\Big|_\circ + \frac{1}{2!}\left(\frac{\partial x}{\partial r}\Delta r + \frac{\partial x}{\partial s}\Delta s + \frac{\partial x}{\partial t}\Delta t + \ldots .\right)^{(2)}\Big|_\circ + \ldots . + \frac{1}{n!}\left(\frac{\partial x}{\partial r}\Delta r + \frac{\partial x}{\partial s}\Delta s + \frac{\partial x}{\partial t}\Delta t + \ldots .\right)^{(n)}\Big|_\circ + R_n\dagger$$

or, $$\Delta x^\circ = \sum_{n=1}^{n} \frac{1}{n!}\left(\frac{\partial x}{\partial r}\Delta r + \frac{\partial x}{\partial s}\Delta s + \frac{\partial x}{\partial t}\Delta t + \ldots .\right)^{(n)}\Big|_\circ + R_n \quad (2)$$

where $\left(\frac{\partial x}{\partial r}\Delta r + \frac{\partial x}{\partial s}\Delta s + \frac{\partial x}{\partial t}\Delta t + \ldots .\right)^{(n)}\Big|_\circ$ is used to express symbolically the n-th order total differential of x, evaluated by means of the initial values, $r^\circ, s^\circ, t^\circ, \ldots .$‡

It is assumed that x is continuous and that all of the derivatives used exist and are continuous. If x has a finite number, n, of derivatives, $R_n = 0$ (Taylor's Formula). If there are an infinite number of derivatives it is assumed that R_n approaches zero, and that the series is absolutely convergent throughout the region considered (Taylor's Series).

*See Art. 132, Higher Mathematics for Engineers and Physicists, by I. S. and E. S. Sokolnikoff.
†Art. 51, Goursat-Hedrick, Math. Analysis, Vol. I; Art. 2, Chap. VII, Osgood, Advanced Calculus.
‡Art. 15, Goursat-Hedrick, Math. Analysis, Vol. I.

Equation (2) may also be written in the following forms:

$$\Delta x^{\circ} = dx\Big|_{\circ} + \frac{d^2x}{2!}\Big|_{\circ} + \ldots . + \frac{d^nx}{n!}\Big|_{\circ} + \ldots . \qquad (3a)$$

and

$$\Delta x^{\circ} = \delta x + \delta^2 x + \ldots . + \delta^n x + \ldots . \qquad (3b)$$

where, by definition, the primary error is $\delta x = dx\Big|_{\circ}$, the secondary error is $\delta^2 x = \frac{d^2x}{2!}\Big|_{\circ}$, etc.

Numerical examples show that Equations (3a) and (3b) may be extended to composite functions, of the type $x(r, s, t, \ldots . ; u, v, \ldots .)$, where $r, s, t, \ldots .$ are functions of the independent variables $u, v, \ldots . .$

III. Primary Errors and Probable Errors

7. *Primary Errors in the Unknowns of Linear Equations.*—Let Equations (1) to (1n), inclusive, in Section 5, represent a system of n linearly independent linear equations in n unknowns, where the coefficients $a_{11} \ldots . a_{nn}$ are integral linear functions of one or more of the p arbitrary constants $k_1 \ldots . k_p$ occurring in the given system, which are subject to the errors or variations $\Delta k_1 \ldots . \Delta k_p$, respectively.

A typical coefficient may be represented by the function $a_{ij}(k_1, \ldots . , k_p)$, whence

$$\Delta a_{ij} = da_{ij}\Big|_{\circ} = \frac{\partial a_{ij}}{\partial k_1}\Delta k_1 + \ldots . + \frac{\partial a_{ij}}{\partial k_p}\Delta k_p \qquad (4)$$

It is assumed that the magnitudes of the errors in the arbitrary constants are so limited that the equations will remain linearly independent (that is, the determinant of the coefficients does not vanish), and that the Taylor's series representing the true error in any of the unknowns will be absolutely convergent.

From Equations (1) to (1n), inclusive, the total differentials $df_1, \ldots . , df_m, \ldots . , df_n$ are as follows:

$$\begin{aligned} df_1 = {} & a_{11}dx_1 + \ldots . + a_{1m}dx_m + \ldots . + a_{1n}dx_n \\ & + x_1da_{11} + \ldots . + x_mda_{1m} + \ldots . + x_nda_{1n} = 0 \end{aligned} \qquad (5)$$

. . . . . .

$$\begin{aligned} df_m = {} & a_{m1}dx_1 + \ldots . + a_{mm}dx_m + \ldots . + a_{mn}dx_n \\ & + x_1da_{m1} + \ldots . + x_mda_{mm} + \ldots . + x_nda_{mn} = 0 \end{aligned} \qquad (5m)$$

. . . . . .

$$df_n = a_{n1}dx_1 + \ldots + a_{nm}dx_m + \ldots + a_{nn}dx_n + x_1da_{n1} + \ldots + x_mda_{nm} + \ldots + x_nda_{nn} = 0 \quad (5n)$$

From these equations the corresponding relations between the errors in the coefficients and the primary errors in the unknowns may be obtained by substitution of the initial values, $a^\circ_{11} \ldots a^\circ_{nn}$ and $x^\circ_1 \ldots x^\circ_n$, as follows:

$$a^\circ_{11}\delta x_1 + \ldots + a^\circ_{1m}\delta x_m + \ldots + a^\circ_{1n}\delta x_n + \sum_{j=1}^{n} x^\circ_j \Delta a_{1j} = 0 \quad (6)$$

. . . . . .

$$a^\circ_{m1}\delta x_1 + \ldots + a^\circ_{mm}\delta x_m + \ldots + a^\circ_{mn}\delta x_n + \sum_{j=1}^{n} x^\circ_j \Delta a_{mj} = 0 \quad (6m)$$

. . . . . .

$$a^\circ_{n1}\delta x_1 + \ldots + a^\circ_{nm}\delta x_m + \ldots + a^\circ_{nn}\delta x_n + \sum_{j=1}^{n} x^\circ_j \Delta a_{nj} = 0 \quad (6n)$$

In solving for the primary errors $\delta x_1 \ldots \delta x_n$, it is desirable, in the first stage, to employ an abbreviated symbol for the terms indicated by the summation sign in the foregoing equations. For this purpose, let

$$e_i = \sum_{j=1}^{n} x^\circ_j \Delta a_{ij} \quad (7)$$

The following equations may then be written in place of Equations (6) to (6n):

$$a^\circ_{11}\delta x_1 + \ldots + a^\circ_{1m}\delta x_m + \ldots + a^\circ_{1n}\delta x_n = -e_1 \quad (8)$$

. . . . . .

$$a^\circ_{m1}\delta x_1 + \ldots + a^\circ_{mm}\delta x_m + \ldots + a^\circ_{mn}\delta x_n = -e_m \quad (8m)$$

. . . . . .

$$a^\circ_{n1}\delta x_1 + \ldots + a^\circ_{nm}\delta x_m + \ldots + a^\circ_{nn}\delta x_n = -e_n \quad (8n)$$

It will be noted that the left-hand sides of these equations differ from the original equations only in containing $\delta x_1 \ldots \delta x_n$ in place of the unknowns $x_1 \ldots x_n$; and that the right-hand terms represent the differential changes produced by the errors or variations in the arbitrary constants of the given equations, multiplied by -1.

A solution of Equations (8) to (8n) will give the desired expressions for the primary errors $\delta x_1 \ldots \delta x_n$, in terms of $e_1 \ldots e_n$. Various methods of obtaining these expressions will be discussed and illustrated in the following sections, before proceeding with the determination of maximum errors.

8. *Solution by Determinants.*—Referring to Equations (1) to (1n), inclusive, Section 5, the determinant of the coefficients, D, is as follows:

$$D = \begin{vmatrix} a^\circ_{11} & \cdots & a^\circ_{1m} & \cdots & a^\circ_{1n} \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ a^\circ_{m1} & \cdots & a^\circ_{mm} & \cdots & a^\circ_{mn} \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ a^\circ_{n1} & \cdots & a^\circ_{nm} & \cdots & a^\circ_{nn} \end{vmatrix} \tag{9a}$$

The initial value of a given unknown, x_m, may be obtained from the relation

$$x^\circ_m = \frac{1}{D} \begin{vmatrix} a^\circ_{11} & \cdots & \mu_1 & \cdots & a^\circ_{1n} \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ a^\circ_{m1} & \cdots & \mu_m & \cdots & a^\circ_{mn} \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ a^\circ_{n1} & \cdots & \mu_n & \cdots & a^\circ_{nn} \end{vmatrix} \tag{10a}$$

Noting now that the determinant of the coefficients of the δx's in Equations (8) to (8n), inclusive, is identical with D, as given by Equation (9a), the following solution is obtained for δx_m:

$$\delta x_m = \frac{1}{-D} \begin{vmatrix} a^\circ_{11} & \cdots & e_1 & \cdots & a^\circ_{1n} \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ a^\circ_{m1} & \cdots & e_m & \cdots & a^\circ_{mn} \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ a^\circ_{n1} & \cdots & e_n & \cdots & a^\circ_{nn} \end{vmatrix} \tag{11a}$$

A comparison of the expression for x°_m, Equation (10a), and the corresponding expression for δx_m, Equation (11a), shows that the expression for δx_m is obtainable by substituting the e terms in place of the μ terms in the former expression and dividing by -1. This suggests the development of the determinants indicated by Equation (10a) with respect to the elements of the m-th column. In order to avoid errors in sign in transforming the expression for x°_m into that for δx_m, it is advisable to preserve the signs of the μ terms as they appear on the right-hand side of the equations. This procedure is facilitated by the use of signed minors, or "co-factors."* For example, in Equations (1) to (1n), inclusive, the coefficient in the i-th row and

*Art. 17, Higher Mathematics for Engineers and Physicists, by I. S. and E. S. Sokolnikoff.

the m-th column of the determinant D is a°_{im}, and its signed minor is $(-1)^{i+m}A_{im} = C_{im}$.

The expressions for D, x°_m, and δx_m may then be written as follows:

$$D = \sum_{i=1}^{n} a_{im}C_{im} \tag{9b}$$

$$x^{\circ}_m = \frac{1}{D}\sum_{i=1}^{n} \mu_i C_{im} \tag{10b}$$

and

$$\delta x_m = \frac{1}{-D}\sum_{i=1}^{n} C_{im}e_i \tag{11b}$$

Example No. 1. It is desired to investigate the effects of errors in the coefficients of the following system of equations, of the type $k_{i1}x + k_{i2}y + k_{i3}z = \mu_i$:

$$2x^{\circ} + 3y^{\circ} + 3z^{\circ} = \quad 12.85 \tag{12a}$$

$$x^{\circ} - 5y^{\circ} + 2z^{\circ} = -13.10 \tag{12b}$$

$$x^{\circ} + 2y^{\circ} - \ z^{\circ} = \quad 8.05 \tag{12c}$$

Using the method of determinants, obtain the expression for the primary errors δx, δy, and δz, in terms of e_1, e_2, and e_3.

Solution: In accordance with Section 7, the following equations may be written:

$$2\delta x + 3\delta y + 3\delta z = -e_1 \tag{13a}$$

$$\delta x - 5\delta y + 2\delta z = -e_2 \tag{13b}$$

$$\delta x + 2\delta y - \ \delta z = -e_3 \tag{13c}$$

where

$$e_1 = x^{\circ}\Delta k_{11} + y^{\circ}\Delta k_{12} + z^{\circ}\Delta k_{13}$$

$$e_2 = x^{\circ}\Delta k_{21} + y^{\circ}\Delta k_{22} + z^{\circ}\Delta k_{23}$$

and

$$e_3 = x^{\circ}\Delta k_{31} + y^{\circ}\Delta k_{32} + z^{\circ}\Delta k_{33}$$

The determinant of the coefficients of Equations (12) and (13) is

$$D = \begin{vmatrix} 2 & 3 & 3 \\ 1 & -5 & 2 \\ 1 & 2 & -1 \end{vmatrix}$$

The development of this determinant with respect to the elements of the first, second, and third columns, respectively, by means of co-factors, is as follows:

$$D_1 = 2(1) + 1(\ \ 9) + 1(\ \ 21) = 32$$

$$D_2 = 3(3) - 5(-5) + 2(-\ \ 1) = 32$$

$$D_3 = 3(7) + 2(-1) - 1(-13) = 32$$

the co-factors being shown in parentheses.

In accordance with Equations (9b) and (10b), the unknowns of Equations (12) are

$$x^\circ = \frac{1}{32}[12.85(1) - 13.10(\ \ 9) + 8.05(\ \ 21)] = \ \ 2.00$$

$$y^\circ = \frac{1}{32}[12.85(3) - 13.10(-5) + 8.05(-\ \ 1)] = \ \ 3.00$$

and $$z^\circ = \frac{1}{32}[12.85(7) - 13.10(-1) + 8.05(-13)] = -0.050$$

The desired expressions for the primary errors in the unknowns, as given by Equation (11b), are then as follows:

$$\delta x = \frac{1}{-32}(\ e_1 + 9e_2 + 21e_3)$$

$$\delta y = \frac{1}{-32}(3e_1 - 5e_2 - \ \ e_3)$$

and $$\delta z = \frac{1}{-32}(7e_1 - \ \ e_2 - 13e_3)$$

which is likewise the solution of Equations (13).

9. *Solution by Elimination.*—In case it is practicable to solve the original equations by the method of elimination, a parallel solution, in which the right-hand terms of the original equations are replaced by $-e_1 \ldots . -e_n$, will give the expression for the primary errors, $\delta x_1 \ldots . \delta x_n$ (see Equations (8)).

Example No. 2. Derive the expression for the primary errors of the unknowns of Example No. 1, in terms of e_1, e_2, and e_3, by solving Equations (13) by the method of elimination:

$$2\delta x + 3\delta y + 3\delta z = -e_1 \tag{13a}$$

$$\delta x - 5\delta y + 2\delta z = -e_2 \tag{13b}$$

$$\delta x + 2\delta y - \delta z = -e_3 \tag{13c}$$

Solution: Elimination of the first unknown from Equations (13a) and (13b), and from Equations (13b) and (13c), results in the equations

$$13\delta y - \delta z = -e_1 + 2e_2 \tag{14a}$$

$$-7\delta y + 3\delta z = -e_2 + e_3 \tag{14b}$$

Elimination of δz from these equations gives

$$\delta y = \frac{1}{32}(-3e_1 + 5e_2 + e_3)$$

Substitution of this value in Equation (14a) gives

$$\delta z = 13\delta y + e_1 - 2e_2 = \frac{1}{32}(-7e_1 + e_2 + 13e_3)$$

Finally, by substituting the foregoing values for δy and δz in Equation (13b), one finds

$$\delta x = -e_2 + 5\delta y - 2\delta z = \frac{1}{32}(-e_1 - 9e_2 - 21e_3)$$

10. *Solution by Means of Unit System* ($\mu_m = -1$).—The coefficients of $e_1 \ldots . e_n$ in the expression for the primary error δx_m in a given unknown x_m° may all be obtained by solving a new system of equations, in which the right-hand term of the m-th equation is placed equal to -1, and the corresponding terms of the remaining equations are zero. For convenience, such a system will be referred to as a "unit system."

The left-hand sides of the equations in the unit system are formed by replacing the elements of each row in the determinant of the coefficients of the original equations by the elements of the corresponding column.

By way of demonstrating this method, consider the following system of equations in three unknowns:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = \mu_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = \mu_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = \mu_3$$

Assuming that it is desired to obtain the expression for the primary error in x_2, the desired expression, in accordance with Equation (11b), is as follows:

$$\delta x_2 = \frac{C_{12}}{-D} e_1 + \frac{C_{22}}{-D} e_2 + \frac{C_{32}}{-D} e_3 \qquad (15)$$

where $C_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$, $C_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$, and $C_{32} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

Consider next the following unit system, formed in the manner specified in the foregoing:

$$a_{11}\bar{x}_1 + a_{21}\bar{x}_2 + a_{31}\bar{x}_3 = 0$$

$$a_{12}\bar{x}_1 + a_{22}\bar{x}_2 + a_{32}\bar{x}_3 = -1$$

$$a_{13}\bar{x}_1 + a_{23}\bar{x}_2 + a_{33}\bar{x}_3 = 0$$

A solution of these equations gives the following values for the unknowns:

$$\bar{x}_1 = \frac{1}{D}\begin{vmatrix} a_{21} & a_{31} \\ a_{23} & a_{33} \end{vmatrix}; \bar{x}_2 = \frac{-1}{D}\begin{vmatrix} a_{11} & a_{31} \\ a_{13} & a_{33} \end{vmatrix}; \text{ and } \bar{x}_3 = \frac{1}{D}\begin{vmatrix} a_{11} & a_{21} \\ a_{13} & a_{23} \end{vmatrix}$$

By referring to Equation (15) it is evident that the foregoing values for $\bar{x}_1 \ldots . \bar{x}_3$ are identical, respectively, with the coefficients of $e_1 \ldots . e_3$ in the expression for δx_2.

This method of obtaining the expression for the primary errors in the unknowns has considerable practical value, inasmuch as it is applicable not only to all ordinary systems of equations, but also to a class of equations of great importance in engineering problems, namely, those which are solvable by successive approximations. In the former case, the corresponding unit systems may be solved by determinants or by elimination; in the latter case, by iteration or by converging increments.*

Another important class of equations, which may be regarded as a special case coming under the present method, are those possessing the property of diagonal symmetry. The Clapeyron equations in mechanics and structural analysis (including the equations of slope and deflection) and the normal equations in Least Squares are generally of this type (see Example No. 11).

Referring to Equations (1) to (1n), inclusive, Section 5, if the initial values of the coefficients are such that the coefficient in the i-th row and the k-th column is the same as that in the k-th row and

*"Solution of Equations in Structural Analysis by Converging Increments," by G. H. Dell, Proc. A.S.C.E., October, 1938.

the i-th column for all values of i and k from 1 to n, a system of diagonally symmetrical equations is obtained.* In determining the primary errors in the unknowns of equations of this type, it is obviously unnecessary to make any re-arrangement of the coefficients when forming the corresponding unit systems.

Example No. 3. Derive the expression for the primary error in x°, Example No. 1, by use of the unit system.

Solution: The unit system is as follows:

$$\begin{aligned} 2\bar{x} + \bar{y} + \bar{z} &= -1 \\ 3\bar{x} - 5\bar{y} + 2\bar{z} &= 0 \\ 3\bar{x} + 2\bar{y} - \bar{z} &= 0 \end{aligned}$$

A solution of these equations by determinants or by elimination gives the following values for the unknowns:

$$\bar{x} = \frac{-1}{32}; \bar{y} = \frac{-9}{32}; \text{and } \bar{z} = \frac{-21}{32}$$

Hence the expression for δx is

$$\delta x = \frac{1}{-32}(e_1 + 9e_2 + 21e_3)$$

11. *Finding the Maximum Primary Error.*—In order to determine the maximum primary error δx_m in a given unknown x_m°, after the expression for δx_m has been obtained in terms of $e_1 \ldots . e_n$ by one of the preceding methods, this expression is expanded in accordance with Equations (7) and (4), Section 7. There is thus obtained an expression in which the errors $\Delta k_1 \ldots . \Delta k_p$ occur linearly. In the case of interdependent coefficients, a re-arrangement of terms is necessary in order to combine the multipliers of a given Δk into a single error-coefficient.

In order to obtain the maximum positive value of the primary error in x_m°, the signs of the increments $\Delta k_1 \ldots . \Delta k_p$ should be the same as those of the corresponding error-coefficients, and, vice versa, in order to obtain the maximum negative value of the primary error in x_m° the signs of the increments $\Delta k_1 \ldots . \Delta k_p$ should be opposite to those of the corresponding error-coefficients. After the signs of

*Note: While, in order to be diagonally symmetrical, the relation $a_{ik}^2 = a_{ki}^2$ must necessarily hold, the coefficients a_{ik} and a_{ki} (considered as variables) are not necessarily interdependent and may become unequal.

the Δk's have been thus determined, by inspection, the numerical value of the maximum primary error is readily calculated.

Example No. 4. Assuming that the coefficients in the equations of Example No. 1 are subject to errors of ± 10 per cent, determine the maximum primary error in $x°$.

Solution: By the methods of Sections 8, 9, and 10, the expression for the primary error in $x°$ was found to be

$$\delta x = \frac{1}{-32}(e_1 + 9e_2 + 21e_3) \tag{16}$$

where

$$e_1 = x°\Delta k_{11} + y°\Delta k_{12} + z°\Delta k_{13}$$

$$e_2 = x°\Delta k_{21} + y°\Delta k_{22} + z°\Delta k_{23}$$

and

$$e_3 = x°\Delta k_{31} + y°\Delta k_{32} + z°\Delta k_{33}$$

Upon substituting the initial values of the unknowns, the expression for the primary error in $x°$ assumes the following expanded form:

$$\delta x = \frac{1}{-32}\begin{bmatrix} 2\Delta k_{11} + 3\Delta k_{12} - 0.05\Delta k_{13} \\ + \ 9(2\Delta k_{21} + 3\Delta k_{22} - 0.05\Delta k_{23}) \\ + 21(2\Delta k_{31} + 3\Delta k_{32} - 0.05\Delta k_{33}) \end{bmatrix}$$

In order to make each term in this expression a positive quantity, the following signs must be used with the various Δk's:

$$\begin{vmatrix} - & - & + \\ - & - & + \\ - & - & + \end{vmatrix}$$

and upon giving the Δk's the specified magnitudes of 10 per cent, the maximum positive value of δx is

$$\delta x \text{ (max.)} = \frac{1}{-32}\begin{bmatrix} 2(-0.2) + 3(-0.3) - 0.05(0.3) \\ + \ 9[2(-0.1) + 3(-0.5) - 0.05(0.2)] \\ + 21[2(-0.1) + 3(-0.2) - 0.05(0.1)] \end{bmatrix}$$

$$= \frac{1}{-32}[-1.315 + 9(-1.710) + 21(-0.805)]$$

$$= +1.050.$$

For a maximum negative primary error, the signs of the Δk's should all be reversed, as follows:

$$\begin{vmatrix} + & + & - \\ + & + & - \\ + & + & - \end{vmatrix}$$

and the maximum negative value of δx is -1.050.

12. *Primary Errors in the Unknowns of Non-linear Equations.*— The following functions will be used to represent a system of n linearly independent non-linear equations in n unknowns:

$$F_1 = b_{11}X_{11} + \ldots + b_{1m}X_{1m} + \ldots + b_{1n}X_{1n} - \nu_1 = 0 \qquad (17)$$

. . . . . .

$$F_m = b_{m1}X_{m1} + \ldots + b_{mm}X_{mm} + \ldots + b_{mn}X_{mn} - \nu_m = 0 \qquad (17m)$$

. . . . . .

$$F_n = b_{n1}X_{n1} + \ldots + b_{nm}X_{nm} + \ldots + b_{nn}X_{nn} - \nu_n = 0 \qquad (17n)$$

In a typical equation, namely,

$$F_i = b_{i1}X_{i1} + \ldots + b_{im}X_{im} + \ldots + b_{in}X_{in} - \nu_i = 0$$

the quantities $X_{i1} \ldots X_{in}$ are functions of one or more of the unknowns $x_1 \ldots x_n$; and the coefficients $b_{i1} \ldots b_{in}$ are functions of one or more of the p arbitrary constants $k_1 \ldots k_p$ occurring in the given system, which are subject to the errors or variations $\Delta k_1 \ldots \Delta k_p$, respectively.

The quantities $X_{i1} \ldots X_{in}$ may be represented by the typical function $X_{ij}(x_1, \ldots, x_n)$, whence

$$\delta X_{ij} = dX_{ij}\Big|_0 = \left(\frac{\partial X_{ij}}{\partial x_1}\delta x_1 + \ldots + \frac{\partial X_{ij}}{\partial x_n}\delta x_n\right)\Big|_0 \qquad (18)$$

A typical coefficient may be represented by the function $b_{ij}(k_1, \ldots, k_p)$, whence

$$\delta b_{ij} = db_{ij}\Big|_0 = \left(\frac{\partial b_{ij}}{\partial k_1}\Delta k_1 + \ldots + \frac{\partial b_{ij}}{\partial k_p}\Delta k_p\right)\Big|_0 \qquad (19)$$

The same assumptions which were made in Section 7 regarding the magnitudes of the errors in the arbitrary constants will again be used. It will also be assumed that the original equations have been solved for the initial values $x_1^\circ \ldots x_n^\circ$.

From Equations (17) to (17n), inclusive, the total differentials $dF_1, \ldots, dF_m, \ldots, dF_n$ are as follows:

$$dF_1 = \frac{\partial F_1}{\partial x_1} dx_1 + \ldots + \frac{\partial F_1}{\partial x_m} dx_m + \ldots + \frac{\partial F_1}{\partial x_n} dx_n + X_{11} db_{11} + \ldots + X_{1m} db_{1m} + \ldots + X_{1n} db_{1n} = 0 \quad (20)$$

. . . . . .

$$dF_m = \frac{\partial F_m}{\partial x_1} dx_1 + \ldots + \frac{\partial F_m}{\partial x_m} dx_m + \ldots + \frac{\partial F_m}{\partial x_n} dx_n + X_{m1} db_{m1} + \ldots + X_{mm} db_{mm} + \ldots + X_{mn} db_{mn} = 0 \quad (20m)$$

. . . . . .

$$dF_n = \frac{\partial F_n}{\partial x_1} dx_1 + \ldots + \frac{\partial F_n}{\partial x_m} dx_m + \ldots + \frac{\partial F_n}{\partial x_n} dx_n + X_{n1} db_{n1} + \ldots + X_{nm} db_{nm} + \ldots + X_{nn} db_{nn} = 0 \quad (20n)$$

From these equations the corresponding relations between the errors in the coefficients and the primary errors in the unknowns may be obtained by substitution of the initial values, $b^\circ_{11} \ldots b^\circ_{nn}$ and $x^\circ_1 \ldots x^\circ_n$, as follows:

$$\left.\frac{\partial F_1}{\partial x_1}\right|_\circ \delta x_1 + \ldots + \left.\frac{\partial F_1}{\partial x_m}\right|_\circ \delta x_m + \ldots + \left.\frac{\partial F_1}{\delta x_n}\right|_\circ \delta x_n + \sum_{j=1}^{n} X^\circ_{1j} \delta b_{1j} = 0 \quad (21)$$

. . . . . .

$$\left.\frac{\partial F_m}{\partial x_1}\right|_\circ \delta x_1 + \ldots + \left.\frac{\partial F_m}{\partial x_m}\right|_\circ \delta x_m + \ldots + \left.\frac{\partial F_m}{\partial x_n}\right|_\circ \delta x_n + \sum_{j=1}^{n} X^\circ_{mj} \delta b_{mj} = 0 \quad (21m)$$

. . . . . .

$$\left.\frac{\partial F_n}{\partial x_1}\right|_\circ \delta x_1 + \ldots + \left.\frac{\partial F_n}{\partial x_m}\right|_\circ \delta x_m + \ldots + \left.\frac{\partial F_n}{\partial x_n}\right|_\circ \delta x_n + \sum_{j=1}^{n} X^\circ_{nj} \delta b_{nj} = 0 \quad (21n)$$

As in the case of linear equations, Section 7, it will be desirable for the present to employ an abbreviated symbol for the terms indicated by the summation sign in Equations (21) to (21n). For this purpose, let

$$E_i = \sum_{j=1}^{n} X^\circ_{ij} \delta b_{ij} \quad (22)$$

The following linear equations may then be written in place of Equations (21):

$$\left.\frac{\partial F_1}{\partial x_1}\right|_o \delta x_1 + \ldots + \left.\frac{\partial F_1}{\partial x_m}\right|_o \delta x_m + \ldots + \left.\frac{\partial F_1}{\partial x_n}\right|_o \delta x_n = -E_1 \quad (21)'$$

. . . . . .

$$\left.\frac{\partial F_m}{\partial x_1}\right|_o \delta x_1 + \ldots + \left.\frac{\partial F_m}{\partial x_m}\right|_o \delta x_m + \ldots + \left.\frac{\partial F_m}{\partial x_n}\right|_o \delta x_n = -E_m \quad (21m)'$$

. . . . . .

$$\left.\frac{\partial F_n}{\partial x_1}\right|_o \delta x_1 + \ldots + \left.\frac{\partial F_n}{\partial x_m}\right|_o \delta x_m + \ldots + \left.\frac{\partial F_n}{\partial x_n}\right|_o \delta x_n = -E_n \quad (21n)'$$

From these equations, the expressions for the primary errors $\delta x_1 \ldots . \delta x_n$ may be obtained by the methods discussed in Sections 8, 9, and 10. It should be noted, however, that the coefficients to be used are those indicated in Equations (21) and (21)′, and not those in the given non-linear equations.

In order to derive working expressions to be used in connection with the method of determinants, it will be noted that the determinant of the coefficients of the unknowns $\delta x_1 \ldots . \delta x_n$ in Equations (21)′ is the Jacobian

$$J = \begin{vmatrix} \left.\frac{\partial F_1}{\partial x_1}\right|_o & \ldots & \left.\frac{\partial F_1}{\partial x_m}\right|_o & \ldots & \left.\frac{\partial F_1}{\partial x_n}\right|_o \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \left.\frac{\partial F_m}{\partial x_1}\right|_o & \ldots & \left.\frac{\partial F_m}{\partial x_m}\right|_o & \ldots & \left.\frac{\partial F_m}{\partial x_n}\right|_o \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \left.\frac{\partial F_n}{\partial x_1}\right|_o & \ldots & \left.\frac{\partial F_n}{\partial x_m}\right|_o & \ldots & \left.\frac{\partial F_n}{\partial x_n}\right|_o \end{vmatrix} \quad (23)$$

The primary error in a given unknown, x_m^o, is then given by the following solution:

$$\delta x_m = \frac{1}{-J} \begin{vmatrix} \left.\frac{\partial F_1}{\partial x_1}\right|_o & \ldots & E_1 & \ldots & \left.\frac{\partial F_1}{\partial x_n}\right|_o \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \left.\frac{\partial F_m}{\partial x_1}\right|_o & \ldots & E_m & \ldots & \left.\frac{\partial F_m}{\partial x_n}\right|_o \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \left.\frac{\partial F_n}{\partial x_1}\right|_o & \ldots & E_n & \ldots & \left.\frac{\partial F_n}{\partial x_n}\right|_o \end{vmatrix} \quad (24a)$$

Denoting the co-factor of the element in the i-th row and the m-th column of the Jacobian by the symbol G_{im}, and developing the foregoing determinant with respect to the elements of the m-th column, the primary error δx_m may be expressed in the following abbreviated form:

$$\delta x_m = \frac{1}{-J} \sum_{i=1}^{n} G_{im} E_i \tag{24b}$$

The routine to be followed in practical problems concerned with the calculation of the primary errors in the unknowns of non-linear equations is quite similar to that described in connection with linear equations. After the coefficients of $E_1 \ldots . E_n$ in the expression for δx_m, Equation (24b), are obtained, the expression is further expanded in accordance with Equations (19) and (22). This procedure results in an expression in which the errors $\Delta k_1 \ldots . \Delta k_p$ occur linearly, and it is thus possible to determine by inspection the proper signs which should accompany the Δk's in order to obtain the maximum primary error in a given unknown.

Example No. 5. The equations of two intersecting curves are as follows:

$$F_1 = k_1 x^2 + k_2 x + k_3 y - 29 = 0 \tag{25}$$

$$F_2 = (k_4 x)^2 + k_4 k_5 xy + (k_5 y)^2 - 102.25 = 0 \tag{26}$$

With the following initial values

$$k_1^\circ = 1,\ k_2^\circ = 2,\ k_3^\circ = -3,\ k_4^\circ = 1.5, \text{ and } k_5^\circ = 2,$$

the original equations are:

$$(x^\circ)^2 + 2x^\circ - 3y^\circ = 29$$

$$2.25(x^\circ)^2 + 3x^\circ y^\circ + 4(y^\circ)^2 = 102.25$$

and the curves intersect in four points, one of which has the values $x^\circ = 5,\ y^\circ = 2$.

It is required to determine the maximum primary error in y° at this point due to possible errors or variations of 10 per cent in the arbitrary constants $k_1 \ldots . k_5$.

Solution: By Equation (23),

$$J = \begin{vmatrix} \left.\frac{\partial F_1}{\partial x}\right|_o & \left.\frac{\partial F_1}{\partial y}\right|_o \\ \left.\frac{\partial F_2}{\partial x}\right|_o & \left.\frac{\partial F_2}{\partial y}\right|_o \end{vmatrix}$$

$$= \begin{vmatrix} 2k_1^\circ x^\circ + k_2^\circ & k_3^\circ \\ 2(k_4^\circ)^2 x^\circ + k_4^\circ k_5^\circ y^\circ & 2(k_5^\circ)^2 y^\circ + k_4^\circ k_5^\circ x^\circ \end{vmatrix}$$

$$= \begin{vmatrix} 12 & -3 \\ 28.5 & 31 \end{vmatrix} = 457.5.$$

By Equation (24a),

$$\delta y = \frac{1}{-J} \begin{vmatrix} \left.\frac{\partial F_1}{\partial x}\right|_o & E_1 \\ \left.\frac{\partial F_2}{\partial x}\right|_o & E_2 \end{vmatrix} = \frac{1}{-J} \begin{vmatrix} 12 & E_1 \\ 28.5 & E_2 \end{vmatrix}$$

or

$$\delta y = \frac{28.5E_1 - 12E_2}{457.5} \tag{27}$$

The values of E_1 and E_2, according to Equations (19) and (22), are the differential changes in F_1 and F_2, respectively, due to variations in the arbitrary constants $k_1 \ldots . k_5$, namely,

$$E_1 = (x^\circ)^2 \Delta k_1 + x^\circ \Delta k_2 + y^\circ \Delta k_3 = 25\Delta k_1 + 5\Delta k_2 + 2\Delta k_3$$

$$E_2 = [2k_4^\circ (x^\circ)^2 + k_5^\circ x^\circ y^\circ]\Delta k_4 + [2k_5^\circ (y^\circ)^2 + k_4^\circ x^\circ y^\circ]\Delta k_5 = 95\Delta k_4 + 31\Delta k_5$$

Referring now to Equation (27), it is seen that, in order to obtain maximum positive δy, the errors in the arbitrary constants should have signs as follows:

$$\Delta k_1 = +0.1 \quad \Delta k_2 = +0.2 \quad \Delta k_3 = +0.3 \quad \Delta k_4 = -0.15 \quad \Delta k_5 = -0.2$$

Upon substituting these values in the foregoing expressions for E_1 and E_2 it is found that $E_1 = 4.1$ and $E_2 = -20.45$; and, from Equation (27),

$$\delta y \text{ (max.)} = \frac{28.5(4.1) + 12(20.45)}{457.5} = +0.792$$

The calculation of the corresponding primary error in x° is as follows:

$$\delta x = \frac{1}{-J}\begin{vmatrix} E_1 & \left.\frac{\partial F_1}{\partial y}\right|_\circ \\ E_2 & \left.\frac{\partial F_2}{\partial y}\right|_\circ \end{vmatrix} = \frac{1}{-J}\begin{vmatrix} E_1 & -3 \\ E_2 & 31 \end{vmatrix}$$

or,
$$\delta x = \frac{31E_1 + 3E_2}{-457.5} = \frac{31(4.1) - 3(20.45)}{-457.5} = -0.144$$

It is thus seen that the coördinates of the point of intersection of the two curves are changed from (5, 2) to (4.856, 2.792) by the assumed errors in the arbitrary constants. Since only the primary errors were considered, the foregoing results should be regarded as approximations.

13. *Effect of Errors in Right-hand Terms.* (*a*) *Linear Equations.*— If, in addition to the errors in the coefficients of Equations (1) to (1n), inclusive, the μ terms are subject to the errors or variations $\Delta\mu_1 \ldots . \Delta\mu_n$, the total differentials in Equations (5) to (5n), inclusive, will include the terms $-d\mu_1 \ldots . -d\mu_n$, and the following relations will be obtained in place of Equations (6) to (6n), inclusive:

$$a^\circ_{11}\delta x_1 + \ldots . + a^\circ_{1m}\delta x_m + \ldots . + a^\circ_{1n}\delta x_n + \sum_{j=1}^{n} x^\circ_j \Delta a_{1j} - \Delta\mu_1 = 0 \quad (28)$$

. . . . . .

$$a^\circ_{m1}\delta x_1 + \ldots . + a^\circ_{mm}\delta x_m + \ldots . + a^\circ_{mn}\delta x_n + \sum_{j=1}^{n} x^\circ_j \Delta a_{mj} - \Delta\mu_m = 0 \quad (28m)$$

. . . . . .

$$a^\circ_{n1}\delta x_1 + \ldots . + a^\circ_{nm}\delta x_m + \ldots . + a^\circ_{nn}\delta x_n + \sum_{j=1}^{n} x^\circ_j \Delta a_{nj} - \Delta\mu_n = 0 \quad (28n)$$

The expression for the primary error in x_m then becomes

$$\delta x_m = \frac{1}{-D}\sum_{i=1}^{n} C_{im}(e_i - \Delta\mu_i) \quad (29a)$$

If the right-hand terms alone are subject to errors or variations, the e terms in the foregoing expression are zero. In this case, $\delta^2 x_m$, ... , $\delta^n x_m$ in Equation (3b), Section 6, are equal to zero, and the primary error and the true error are identical, namely,

$$\Delta x_m^\circ = \delta x_m = \frac{1}{D}\sum_{i=1}^{n} C_{im}\Delta\mu_i \qquad (29b)$$

(b) *Non-linear Equations.* If, in addition to the errors in the coefficients of Equations (17) to (17n), inclusive, the ν terms are subject to the errors or variations $\Delta\nu_1 \ldots . \Delta\nu_n$, the total differentials in Equations (20) to (20n), inclusive, will include the terms $-d\nu_1 \ldots . -d\nu_n$. The ν terms may consist of one or more arbitrary constants occurring linearly or non-linearly. In any case, the true maximum errors $\Delta\nu_1 \ldots . \Delta\nu_n$ may be calculated.

The following relations are then obtained in place of Equations (21) to (21n), inclusive,

$$\frac{\partial F_1}{\partial x_1}\Big|_o \delta x_1 + \ldots . + \frac{\partial F_1}{\partial x_m}\Big|_o \delta x_m + \ldots . + \frac{\partial F_1}{\partial x_n}\Big|_o \delta x_n + \sum_{j=1}^{n} X_{1j}^\circ \delta b_{1j} - \Delta\nu_1 = 0 \quad (30)$$

. . . . . .

$$\frac{\partial F_m}{\partial x_1}\Big|_o \delta x_1 + \ldots . + \frac{\partial F_m}{\partial x_m}\Big|_o \delta x_m + \ldots . + \frac{\partial F_m}{\partial x_n}\Big|_o \delta x_n + \sum_{j=1}^{n} X_{mj}^\circ \delta b_{mj} - \Delta\nu_m = 0 \quad (30m)$$

. . . . . .

$$\frac{\partial F_n}{\partial x_1}\Big|_o \delta x_1 + \ldots . + \frac{\partial F_n}{\partial x_m}\Big|_o \delta x_m + \ldots . + \frac{\partial F_n}{\partial x_n}\Big|_o \delta x_n + \sum_{j=1}^{n} X_{nj}^\circ \delta b_{nj} - \Delta\nu_n = 0 \quad (30n)$$

The expression for the primary error in x_m is then as follows:

$$\delta x_m = \frac{1}{-J}\sum_{i=1}^{n} G_{im}(E_i - \Delta\nu_i) \qquad (31a)$$

If the ν terms alone are subject to errors or variations, the E terms in Equation (31a) will be zero, and the expression for the primary error becomes

$$\delta x_m = \frac{1}{J}\sum_{i=1}^{n} G_{im}\Delta\nu_i \qquad (31b)$$

An important use of the relations expressed by Equation (31b) occurs in connection with the solution of non-linear equations, as will be explained in Section 16.

Example No. 6. In connection with the equations of Example No. 5, find the maximum primary error in y° and the corresponding primary error in x°, due to errors of 10 per cent in the right-hand terms.

Solution: By Equation (31b),

$$\delta y = \frac{\begin{vmatrix} 12 & \Delta\nu_1 \\ 28.5 & \Delta\nu_2 \end{vmatrix}}{\begin{vmatrix} 12 & -3 \\ 28.5 & 31 \end{vmatrix}} = \frac{-28.5\Delta\nu_1 + 12\Delta\nu_2}{457.5}$$

$$\delta x = \frac{\begin{vmatrix} \Delta\nu_1 & -3 \\ \Delta\nu_2 & 31 \end{vmatrix}}{\begin{vmatrix} 12 & -3 \\ 28.5 & 31 \end{vmatrix}} = \frac{31\Delta\nu_1 + 3\Delta\nu_2}{457.5}$$

In order to obtain the maximum primary error in y°, $\Delta\nu_1$ and $\Delta\nu_2$ should have opposite signs. Hence

$$\delta y \text{ (max.)} = \pm\frac{28.5\,(2.9) + 12\,(10.225)}{457.5} = \pm 0.45, \qquad \text{and}$$

$$\delta x = \mp\frac{31\,(2.9) - 3\,(10.225)}{457.5} = \mp 0.13.$$

14. *Partial Differentials and Partial Derivatives.*—It has been seen that the expression for the primary error of a given unknown, in the case of both linear and non-linear equations, can be put in the following form:

$$\delta x_m = c_1^\circ\,\Delta k_1 + \ldots + c_p^\circ\,\Delta k_p \tag{32}$$

or, using the differential notation,

$$dx_m = c_1\,dk_1 + \ldots + c_p\,dk_p \tag{33}$$

where dx_m is defined as follows:

$$dx_m = \frac{\partial x_m}{\partial k_1}\,dk_1 + \ldots + \frac{\partial x_m}{\partial k_p}\,dk_p \tag{34}$$

In Equation (32), the coefficients of $\Delta k_1 \ldots \Delta k_p$, which will be designated as "error-coefficients," are functions of the initial values $k_1^\circ \ldots k_p^\circ$, and $x_1^\circ \ldots x_n^\circ$.

The relation expressed by Equation (33) is a more general one, in which the error-coefficients $c_1 \ldots c_p$ are functions of any values of $k_1 \ldots k_p$ and $x_1 \ldots x_n$ which are satisfied by Equations (1) to (1n), inclusive, or by Equations (17) to (17n), inclusive. A comparison of Equations (33) and (34) shows that the terms $c_1\,dk_1 \ldots$

$c_p\,dk_p$ are the partial differentials $\frac{\partial x_m}{\partial k_1}\,dk_1 \ldots . \frac{\partial x_m}{\partial k_p}\,dk_p$, respectively; also that the error-coefficients $c_1 \ldots . c_p$ are the partial derivatives $\frac{\partial x_m}{\partial k_1} \ldots . \frac{\partial x_m}{\partial k_p}$, respectively.

In Section 17 it will be shown that the determination of the true maximum error involves the investigation of the signs of these partial derivatives for various values of the arbitrary constants (and for the corresponding values of the unknowns).

In regard to the effects of errors or variations in an individual arbitrary constant, k_i, of a system of simultaneous equations, the primary error in x_m is given by $\left.\frac{\partial x_m}{\partial k_i}\right|_o \Delta k_i$. Using initial values, Equation (34) may be written as follows:

$$\delta x_m = \left.\frac{\partial x_m}{\partial k_1}\right|_o \Delta k_1 + \ldots . + \left.\frac{\partial x_m}{\partial k_p}\right|_o \Delta k_p \qquad (35)$$

from which it is seen, upon comparison with Equation (32), that the primary error $\left.\frac{\partial x_m}{\partial k_i}\right|_o \Delta k_i$ caused by a variation in an individual arbitrary constant may be obtained from the general expressions which were derived in Sections 8 and 12 by placing the remaining Δk's equal to zero.

15. *Probable Errors.*—In the preceding section, considering the unknowns $x_1 \ldots . x_n$ of a system of simultaneous equations as functions of the arbitrary constants, it was pointed out that the primary error δx_m is equivalent to the sum of the primary errors $\left.\frac{\partial x_m}{\partial k_1}\right|_o \Delta k_1 \ldots . \left.\frac{\partial x_m}{\partial k_p}\right|_o \Delta k_p$, obtained by allowing $k_1 \ldots . k_p$ to vary one at a time.

In a similar manner, if the arbitrary constants $k_1 \ldots . k_p$ are subject to the probable errors $\pm\Delta k_1 \ldots . \pm\Delta k_p$, the corresponding probable errors in a given unknown, x_m^o, are $\pm\left.\frac{\partial x_m}{\partial k_1}\right|_o \Delta k_1, \ldots . , \pm\left.\frac{\partial x_m}{\partial k_p}\right|_o \Delta k_p$, respectively.

The probable error in x_m^o, due to these combined effects, is found by taking the square root of the sum of the squares of the individual

effects;* or, denoting the probable error in x_m° by the symbol λx_m°,

$$\lambda x_m^\circ = \pm\sqrt{\left(\frac{\partial x_m}{\partial k_1}\bigg|_n \Delta k_1\right)^2 + \ldots + \left(\frac{\partial x_m}{\partial k_p}\bigg|_o \Delta k_p\right)^2} \qquad (36)$$

It will be noted that the application of Equation (36) is facilitated by first writing the expression for the primary error, as given by Equations (11b) and (24b), and then reducing to the form indicated by Equation (32).

If the effects of probable errors in the right-hand terms are to be included in the calculation of the probable error in x_m°, Equation (36) should be extended by including under the radical the sum of the squares of the corresponding terms occurring in the expression for δx_m according to Section 13 (see Example No. 8).

Example No. 7. Let it be required to calculate the probable error in x° of Example No. 4, assuming the arbitrary constants to contain probable errors amounting to ± 5 per cent.

Solution: In Example No. 4, the expression for δx was found to be as follows:

$$\delta x = \frac{1}{-32}\begin{bmatrix} (2\Delta k_{11} + 3\Delta k_{12} - 0.05\Delta k_{13}) \\ + \ 9(2\Delta k_{21} + 3\Delta k_{22} - 0.05\Delta k_{23}) \\ + 21(2\Delta k_{31} + 3\Delta k_{32} - 0.05\Delta k_{33}) \end{bmatrix}$$

In accordance with Equation (36), the probable error in x° is then

$$\lambda x^\circ = \pm\frac{1}{32}\sqrt{\begin{matrix} (0.2)^2 + (0.45)^2 + (0.0075)^2 \\ + \ 81[(0.1)^2 + (0.75)^2 + (0.005)^2] \\ + 441[(0.1)^2 + (0.3)^2 + (0.0025)^2] \end{matrix}} = \pm 0.298$$

Example No. 8. Find the probable error in y° of the equations in Examples No. 5 and No. 6, assuming that the arbitrary constants and also the right-hand terms are subject to probable errors amounting to ± 5 per cent.

Solution: In Example No. 5, the primary error in y° due to errors in the arbitrary constants of the coefficients was as follows:

$$\delta y = \frac{1}{-457.5}\begin{bmatrix} -28.5(25\Delta k_1 + 5\Delta k_2 + 2\Delta k_3) \\ \\ +12(95\Delta k_4 + 31\Delta k_5) \end{bmatrix}$$

*Art. 3, Chap. III, Geodesy, by A. R. Clarke.

The primary error in y° due to errors in the right-hand terms, as shown in Example No. 6, is

$$\delta y = \frac{1}{-457.5}(28.5\Delta\nu_1 - 12\Delta\nu_2)$$

Upon substituting the assumed values of $\Delta k_1 \ldots . \Delta k_5$ and of $\Delta\nu_1$ and $\Delta\nu_2$ and proceeding in accordance with Equation (36), the following computation gives the required probable error in y°:

$$\lambda y^\circ = \pm\frac{1}{457.5}\sqrt{\begin{matrix}812.25[(1.25)^2 + (0.5)^2 + (0.3)^2] \\ + 144[(7.125)^2 + (3.1)^2] \\ + 812.25(1.45)^2 + 144(5.1125)^2\end{matrix}} = \pm 0.274$$

IV. True Errors

16. *Methods Available for Calculating the True Error.*—The true error in the unknowns of a system of simultaneous equations, due to errors of definite magnitude and sign in the arbitrary constants, can be calculated in three ways, as follows:

(1) By writing and solving the revised equations.

(2) By using approximate values for the unknowns in the revised equations and then varying the right-hand terms and calculating the corresponding corrections to the unknowns (as in Newton's Method).

(3) By Taylor's series, involving the calculation of the errors of higher orders.

These methods are all of value in investigating the true maximum error. The first is best adapted to linear equations, and the second to non-linear equations. The third method is more limited in its usefulness as a tool for numerical calculation, but has considerable analytical value. A brief explanation of the various methods follows.

First Method. Upon substitution of the modified coefficients, $a'_{11} \ldots . a'_{nn}$, in Equations (1 to (1n), inclusive, the revised equations are

$$a'_{11}x'_1 + \ldots . + a'_{1m}x'_m + \ldots . + a'_{1n}x'_n - \mu^\circ_1 = 0 \qquad (37)$$

. . . . . .

$$a'_{m1}x'_1 + \ldots . + a'_{mm}x'_m + \ldots . + a'_{mn}x'_n - \mu^\circ_m = 0 \qquad (37m)$$

. . . . . .

$$a'_{n1}x'_1 + \ldots . + a'_{nm}x'_m + \ldots . + a'_{nn}x'_n - \mu^\circ_n = 0 \qquad (37n)$$

The true error in a given unknown, x_m°, is then given by $\Delta x_m = x'_m - x_m^\circ$, where x'_m is obtained by solving the revised equations, and x_m° is the initial value of the unknown.

Second Method. In considering the application of this method to linear equations, let it be supposed that the unknowns $x'_1 \ldots . x'_n$ in Equations (37) to (37n) are approximately given by $x_1 \ldots . x_n$, respectively. It is generally convenient to make use of the primary errors in obtaining such approximate values, in which case $x_1 = x_1^\circ + \delta x_1$, $x_2 = x_2^\circ + \delta x_2$, etc. Next, suppose that values of $\mu'_1 \ldots . \mu'_n$ are computed from the relations,

$$a'_{11}x_1 + \ldots . + a'_{1m}x_m + \ldots . + a'_{1n}x_n = \mu'_1 \qquad (38)$$

. . . . . .

$$a'_{m1}x_1 + \ldots . + a'_{mm}x_m + \ldots . + a'_{mn}x_n = \mu'_m \qquad (38m)$$

. . . . . .

$$a'_{n1}x_1 + \ldots . + a'_{nm}x_m + \ldots . + a'_{nn}x_n = \mu'_n \qquad (38n)$$

If the right-hand terms of Equations (38) to (38n) are now given the variations $\Delta\mu_1 = \mu_1^\circ - \mu'_1$, $\Delta\mu_2 = \mu_2^\circ - \mu'_2$, etc., the method described in Section 13 may be used to calculate the corresponding variations $\Delta' x_1 \ldots . \Delta' x_n$ in the approximate values of the unknowns, $x_1 \ldots . x_n$; and, as stated in Section 13, these variations will be the true errors, or true corrections, when dealing with linear equations. The true error in x_m° is then $\Delta x_m^\circ = \delta x_m + \Delta' x_m$.

The application of this method to non-linear equations is similar to that explained in the foregoing, with the exception that when the right-hand terms $\nu'_1 \ldots . \nu'_n$ are given the variations $\Delta\nu_1 = \nu_1^\circ - \nu'_1$, $\Delta\nu_2 = \nu_2^\circ - \nu'_2$, etc., the corresponding variations obtained in accordance with Equation (31b), Section 13, are not the true errors in the unknowns; consequently, the process is repeated until the corrections to the unknowns become negligible.

It will be noted that this procedure affords a convenient method of solving non-linear equations after approximate values of the unknowns have been determined.

Example No. 10 illustrates the use of this method in connection with non-linear equations.

Third Method. If the expression for the primary error of a given unknown, x_m°, of a system of *linear* equations is changed in such a way as to replace the unknowns $x_1^\circ \ldots . x_n^\circ$ by the primary errors $\delta x_1 \ldots . \delta x_n$, respectively, the resulting expression represents the value of the secondary error, $\delta^2 x_m$. Likewise, if in this expression for the

secondary error, the quantities $\delta x_1 \ldots . \delta x_n$ are replaced by $\delta^2 x_1 \ldots .$ $\delta^2 x_n$, respectively, the resulting expression represents the value of the third order error, $\delta^3 x_m$. In a similar way, one may obtain each term in the Taylor's series, Equation (3b), Section 6, after the errors of the next lower order have been calculated. The method of deriving these results is given in the Appendix.

This method of calculating true errors appears to be chiefly useful in studying the effect of variations in individual arbitrary constants, or in small groups thereof, in connection with the determination of the true maximum error.

The expressions for the errors of higher order in the unknowns of non-linear equations become increasingly complicated, and are therefore of insufficient practical value to warrant their inclusion in the present treatment.

17. *The True Maximum Error.*—In investigating the true maximum error in a given unknown of a system of simultaneous equations when the assumed errors or variations $\Delta k_1 \ldots . \Delta k_p$ in the arbitrary constants $k_1 \ldots . k_p$, respectively, are assigned definite limits, the usual concern is the determination of the correct signs to accompany the Δk's. In practical engineering problems, the true maximum error requires, as a general rule, the use of the assumed maximum value for each of the Δk's; in other words, a true extremum, as treated by the methods of maxima and minima, will seldom be a condition affecting the true maximum error. Consequently, the present discussion will deal principally with cases which are susceptible to treatment by elementary methods.

Since a given unknown x_m of a system of linearly independent equations is fully determined for definite values of the arbitrary constants, it will be convenient to consider x_m as a function of the independent variables, $k_1 \ldots . k_p$, as follows:

$$x_m = f(k_1, \ldots . , k_p) \tag{39}$$

The true maximum error will be obtained by making $|x_m - x_m^\circ|$ as great as possible without permitting the Δk's to exceed their assumed maximum values.

Let $\bar{x}_m - x_m^\circ$ denote the maximum positive error in x_m°, and let $\underline{x}_m - x_m^\circ$ denote the maximum negative error. Then, by means of appropriate variations in the magnitudes of the Δk's, x_m may be made to vary continuously throughout the region $\underline{x}_m \leqq x_m \leqq \bar{x}_m$. As the ensuing arguments will apply equally well to the maximum positive

error and to the maximum negative error, it will be sufficient to confine the discussion to the region $x_m^\circ \gtrless x_m \gtrless \bar{x}_m$.

The procedure that will be used in arriving at the true maximum error may be outlined as follows: Using the assumed maximum values of the Δk's in conjunction with such signs as appear most likely to lead to the maximum error in x_m°, the true values of the unknowns, $x_1' \ldots . x_n'$, are calculated. In this way a point x_m' in the region under consideration is obtained, and the signs of the partial derivatives $\frac{\partial x_m}{\partial k_1} \ldots . \frac{\partial x_m}{\partial k_p}$ are then investigated. If these signs are such that a further increase in $x_m - x_m^\circ$ can be obtained by increasing each of the Δk's beyond their assumed maximum values, the difference $x_m' - x_m^\circ$ is the true maximum error. If, on the contrary, the signs of some of the partial derivatives, say $\frac{\partial x_m}{\partial k_h} \ldots . \frac{\partial x_m}{\partial k_k}$ at this point are such that a decrease in the absolute value of the corresponding Δk's will produce a further increase in $x_m - x_m^\circ$, the signs of $\Delta k_h \ldots . \Delta k_k$ are reversed, and a new set of modified unknowns, $x_1'' \ldots . x_n''$, are calculated. The signs of all of the partial derivatives $\frac{\partial x_m}{\partial k_1} \ldots . \frac{\partial x_m}{\partial k_p}$ are again investigated; and by applying the criteria previously described it is learned whether or not the difference $x_m'' - x_m^\circ$ is a maximum. If the condition of an extremum is not involved, the true maximum error will be obtainable by means of the simple process described.

If an extremum should be involved, or, in other words, if the maximum error occurs when for some of the arbitrary constants the partial derivatives $\frac{\partial x_m}{\partial k}$ have zero values, it becomes necessary, in general, to use values for the corresponding Δk's which are less than the assumed maximum values. In such cases, it is still necessary to use the assumed maximum values of the Δk's for those arbitrary constants for which $\frac{\partial x_m}{\partial k}$ does not vanish.

The process of determining the signs of the partial derivatives $\frac{\partial x_m}{\partial k_1} \ldots . \frac{\partial x_m}{\partial k_p}$ at the point x_m' is most readily performed by taking the expression for δx_m (the primary error), and substituting therein the modified values of the arbitrary constants $(k_1' \ldots . k_p')$ and of the unknowns $(x_1' \ldots . x_n')$ in place of the corresponding initial values.

Such an expression will, for convenience, be referred to as the "criterion expression," or "the expression for $\delta x'_m$." The error-coefficients in the criterion expression will be the values of the partial derivatives $\frac{\partial x_m}{\partial k_1} \ldots . \frac{\partial x_m}{\partial k_p}$ at the point in question.

A brief examination will now be made into the reasoning underlying the procedure which has been outlined for finding the true maximum error in x_m°. Referring to the function indicated by Equation (39), it will be assumed that x_m° has been changed to another point, x_m, in the region $x_m^\circ \lessgtr x_m \lessgtr \bar{x}_m$, by means of certain increments $\Delta k_1 \ldots . \Delta k_p$ which have been given such signs as are required for a maximum positive primary error, δx_m. If the values of the Δk's are taken sufficiently small, the signs of the partial derivatives $\frac{\partial x_m}{\partial k_1} \ldots . \frac{\partial x_m}{\partial k_p}$ will remain unchanged; accordingly, in order to secure a further increase in the difference $x_m - x_m^\circ$, that is, in the true error, Δx_m, it is sufficient to increase the value of each of the increments $\Delta k_1 \ldots . \Delta k_p$ by a small amount. Furthermore, in the case of every arbitrary constant for which $\frac{\partial x_m}{\partial k_i}$ remains unchanged in sign throughout the interval $x_m^\circ \lessgtr x_m \lessgtr \bar{x}_m$, it is necessary to use the assumed maximum value of Δk_i in order to make the difference $x_m - x_m^\circ$ as large as possible.

Suppose now, following the procedure outlined earlier, that the unknowns $x_1^\circ \ldots . x_n^\circ$ have been changed to $x'_1 \ldots . x'_n$, respectively, by using the assumed maximum values of the Δk's in conjunction with the signs required for making the primary error, δx_m, a maximum. From the preceding discussion it is seen that if none of the partial derivatives $\frac{\partial x_m}{\partial k_1} \ldots . \frac{\partial x_m}{\partial k_p}$ have undergone a change of sign as x_m varied from x_m° to x'_m, the maximum error in x_m° is $x'_m - x_m^\circ = \bar{x}_m - x_m^\circ$.

Let us now consider the case where some (but not all) of the partial derivatives, say, $\frac{\partial x_m}{\partial k_h} \ldots . \frac{\partial x_m}{\partial k_k}$, have changed sign while x_m changed from x_m° to x'_m. Up to the present time, those Δk's for which the partial derivatives have undergone no change in sign have rendered their full contribution towards obtaining a maximum error in x_m°, and it will be advantageous, therefore, to treat these arbitrary constants as ordinary fixed constants while investigating further changes in the remaining arbitrary constants, and to consider the function

$$x_m = g(k_h, \ldots . , k_k)$$

in the region $x'_m \gtrless x_m \gtrless \bar{x}_m$. Since the signs of the partial derivatives $\frac{\partial x_m}{\partial k_h} \ldots \frac{\partial x_m}{\partial k_k}$ at the point x'_m are now assumed to be opposite to those of the error-coefficients in the expression for the primary error in x°_m, the value of $x_m - x^\circ_m$ will be further increased if the increments $\Delta k_h \ldots \Delta k_k$, each or all, are reduced by a sufficiently small amount from their assumed maximum values. Furthermore, if these partial derivatives undergo no further change in sign while $\Delta k_h \ldots \Delta k_k$ are gradually reduced to zero, the value of $x_m - x^\circ_m$ will continue to increase. Beyond this stage, by reversing the signs of the Δk's in question, and assigning them sufficiently small values, it will be possible to cause the difference $x_m - x^\circ_m$ to increase still further; and if, as the Δk's approach their assumed maximum values (their signs having been reversed) the signs of the partial derivatives $\frac{\partial x_m}{\partial k_h} \ldots \frac{\partial x_m}{\partial k_k}$ do not again change, the maximum error will be obtained by using the assumed maximum values of $\Delta k_h \ldots \Delta k_k$.

It is thus seen that if, following the recommended procedure, a new set of modified unknowns, $x''_1 \ldots x''_n$, is calculated after reversing the signs of $\Delta k_h \ldots \Delta k_k$ (whose magnitudes are taken equal to their assumed maximum values), and the partial derivatives $\frac{\partial x_m}{\partial k_1} \ldots \frac{\partial x_m}{\partial k_p}$ are all of such sign, at the point x''_m, that a further increase in $x_m - x^\circ_m$ will be obtained by increasing each of the Δk's beyond their assumed maximum values, the true maximum error is given by $x''_m - x^\circ_m = \bar{x}_m - x^\circ_m$.

It may be found necessary in some instances to repeat the above process for a third set of unknowns $x'''_1 \ldots x'''_m$, before the true maximum error can be found.

In some problems it is possible to foresee a change in sign of some of the partial derivatives $\frac{\partial x_m}{\partial k_1} \ldots \frac{\partial x_m}{\partial k_p}$ before the true values of $x'_1 \ldots x'_n$ are calculated. For example, if, when signs are used with the Δk's as required to obtain a maximum primary error in x°_m, approximate values of $x'_1 \ldots x'_n$ are found from a computation of the primary errors $\delta x_1 \ldots \delta x_n$, and if it is thereby indicated that some of the unknowns may change sign, it is seen, by referring to Equation (11b), that the signs of some of the quantities $x' \Delta a_{ij}$ in the corresponding criterion expression may likewise change. It is therefore advisable, before trying for a true maximum error, to calcu-

late the primary errors in all of the unknowns, and, if it appears probable that some of the error-coefficients will change sign, to reverse the signs of the appropriate Δk's before computing the apparent maximum error.

Finally, in regard to cases involving an extremum, it may be said that if, after the difference $x_m - x_m^\circ$ has been made as large as possible by using the assumed maximum values of the Δk's with those arbitrary constants for which the partial derivatives $\frac{\partial x_m}{\partial k}$ do not vanish, there remain some arbitrary constants for which the partial derivatives $\frac{\partial x_m}{\partial k_i}$ do vanish when the absolute values of the corresponding Δk's are less than their assumed maximum values, the methods of maxima and minima must ordinarily be used to determine the exact values which those Δk's require in order to secure the true maximum error. It is difficult to formulate definite rules setting forth a general procedure for dealing with such cases without expanding the present study far beyond the limits contemplated. However, it will be observed, first, that the procedure previously described enables one to determine certain regions in the neighborhood of $\bar{x}_m$ in which the possible existence of an extremum will be indicated by a change in sign of $\frac{\partial x_m}{\partial k_i}$ when the sign of the accompanying Δk is reversed; and, second, to determine for which of the arbitrary constants the assumed maximum values of the Δk's should be used.

Example No. 9. From the equations used in Examples Nos. 1 and 4, let it be required to find the true maximum positive and negative errors in x° due to the assumed maximum errors of 10 per cent in the arbitrary constants.

Since there are nine arbitrary constants in the given system, the total number of ways in which the signs of the specified errors can be combined is equal to $2^9 = 512$. The problem involves finding two particular combinations, namely, those required for producing the maximum positive error and the maximum negative error, respectively, in x°.

The signs to be taken in computing the apparent maximum error in x° are those which are required to render the primary error a maximum, unless it can be foreseen that some of the error-coefficients $\frac{\partial x}{\partial k_{11}}$, $\frac{\partial x}{\partial k_{12}}$, etc., undergo a change in sign while x is changing from

x° to x'. This will be the case if the sign of x', y', or z' is different from that of x°, y°, or z°, respectively. In this event, for purposes of computing the apparent maximum error, the signs of those Δk's whose coefficients contain the unknown which has changed from positive to negative, or vice versa, should be taken opposite to those required by them for making the primary error in x° a maximum. Therefore, in order to make a more intelligent estimate of the signs required for the true maximum error, it is expedient to first calculate the primary errors in all of the unknowns, as in this way it is generally possible to determine whether a change in sign occurs in any of the unknowns.

The expressions for δx, δy, and δz were found to be as follows:

$$\delta x = \frac{1}{-32}(\ e_1 + 9e_2 + 21e_3)$$

$$\delta y = \frac{1}{-32}(3e_1 - 5e_2 - \quad e_3)$$

$$\delta z = \frac{1}{-32}(7e_1 - \quad e_2 - 13e_3)$$

where, for maximum positive δx the values of e_1, e_2, and e_3 were found to be -1.315, -1.710, and -0.805, respectively. Hence, when δx (max.) $= +1.050$

$$\delta y = \frac{1}{-32}(-3.945 + 8.550 + \ 0.805) = -0.169$$

and

$$\delta z = \frac{1}{-32}(-9.205 + 1.710 + 10.465) = -0.093$$

The modified values of the unknowns will be approximately as follows:

$$x' \doteq 2.000 + 1.050 = 3.050$$

$$y' \doteq 3.000 - 0.169 = 2.831$$

$$z' \doteq -0.050 - 0.093 = -0.143$$

therefore, since none of the unknowns have changed sign, the apparent maximum error will be calculated by using with the Δk's the same set of signs as was used in computing δx, namely,

$$\begin{vmatrix} - & - & + \\ - & - & + \\ - & - & + \end{vmatrix}$$

The revised equations are then

$$1.8x' + 2.7y' + 3.3z' = \quad 12.85$$
$$0.9x' - 5.5y' + 2.2z' = -13.10$$
$$0.9x' + 1.8y' - 0.9z' = \quad 8.05$$

for which the solution is as follows:

$$D' = \quad 1.8(0.99) + 0.9\ (\ 8.37) + 0.9\ (\ 24.09) = 30.996$$

$$x' = \frac{1}{30.996}[12.85(0.99) - 13.10(\ 8.37) + 8.05(\ 24.09)] = \quad 3.129$$

$$y' = \frac{1}{30.996}[12.85(2.79) - 13.10(-4.59) + 8.05(-\ 0.99)] = \quad 2.839$$

$$z' = \frac{1}{30.996}[12.85(6.57) - 13.10(-0.81) + 8.05(-12.33)] = -0.1362$$

The apparent maximum error (positive) in x° is therefore $3.129 - 2.00 = +1.129$.

Test: As previously stated, in order to determine whether or not the foregoing result is the true maximum (positive) error in x°, it is necessary to see whether or not the products, or differentials, $\frac{\partial x'}{\partial k_{11}}\Delta k_{11}$, $\frac{\partial x'}{\partial k_{12}}\Delta k_{12}$, etc., are all of like sign. This process is equivalent to taking the revised equations and writing out the expression for the primary error in x', and investigating the resulting signs of the separate terms, using for the Δk's the same signs as were employed in the revised equations from which the apparent maximum error was computed. As previously observed, the expression for the primary error in x' is obtained from the determinant solution of x' by substituting e_1', e_2', and e_3' in place of the μ values, 12.85, -13.10 and 8.05, respectively, and dividing by -1; hence,

$$\delta x' = \frac{1}{-30.996}(0.99e_1' + 8.37e_2' + 24.09e_3')$$

where

$$e_1' = 3.129\,\Delta k_{11} + 2.839\,\Delta k_{12} - 0.1362\,\Delta k_{13}$$
$$e_2' = 3.129\,\Delta k_{21} + 2.839\,\Delta k_{22} - 0.1362\,\Delta k_{23}$$
$$e_3' = 3.129\,\Delta k_{31} + 2.839\,\Delta k_{32} - 0.1362\,\Delta k_{33}$$

It is seen that each of the nine terms in the foregoing expression is positive, and therefore the apparent maximum error (positive) is also the true maximum error, namely $\Delta x^\circ = +1.129$.

Maximum Negative Δx°. As previously shown, the expression for the primary error in x° was

$$\delta x = \frac{1}{-32}\begin{bmatrix} 2\Delta k_{11} + 3\Delta k_{12} - 0.05\Delta k_{13} \\ + 9(2\Delta k_{21} + 3\Delta k_{22} - 0.05\Delta k_{23}) \\ + 21(2\Delta k_{31} + 3\Delta k_{32} - 0.05\Delta k_{33}) \end{bmatrix}$$

which, in order to yield its maximum negative value, requires the following signs:

$$\begin{vmatrix} + & + & - \\ + & + & - \\ + & + & - \end{vmatrix}$$

The primary errors obtained by the use of these signs are δx (max.) $= -1.050$, $\delta y = +0.169$, and $\delta z = +0.093$.

The modified values of the unknowns will be approximately as follows:

$$x' \doteq 2.000 - 1.050 = 0.950$$

$$y' \doteq 3.000 + 0.169 = 3.169$$

$$z' \doteq -0.050 + 0.093 = 0.043$$

According to these indications, a change in sign occurs in z while x is changing from x° to x'. Therefore, in solving for a tentative, or apparent, maximum error, the signs given in the foregoing as being required for obtaining the maximum primary error will be modified by reversing the signs of those Δk's which are multiplied by this unknown (z), as follows:

$$\begin{vmatrix} + & + & + \\ + & + & + \\ + & + & + \end{vmatrix}$$

The revised equations are then

$$2.2x' + 3.3y' + 3.3z' = 12.85$$

$$1.1x' - 4.5y' + 2.2z' = -13.10$$

$$1.1x' + 2.2y' - 0.9z' = 8.05$$

The solution of the revised equations is as follows:

$$D' = \quad 2.2\ (-0.79) + 1.1\ (\ 10.23) + 1.1(\ \ 22.11) = 33.836$$

$$x' = \frac{1}{33.836}[12.85(-0.79) - 13.10(\ 10.23) + 8.05(\ \ 22.11)] = \ 1.000$$

$$y' = \frac{1}{33.836}[12.85(\ \ 3.41) - 13.10(-5.61) + 8.05(-\ \ 1.21)] = \ 3.179$$

$$z' = \frac{1}{33.836}[12.85(\ \ 7.37) - 13.10(-1.21) + 8.05(-13.53)] = 0.0484$$

The apparent maximum error in x° is, therefore, $1.000 - 2.000 = -1.000$

Test: $$\delta x' = \frac{1}{-33.836}(-0.79e_1' + 10.23e_2' + 22.11e_3')$$

where $$e_i' = 1.000\Delta k_{i1} + 3.179\Delta k_{i2} + 0.0484\Delta k_{i3}$$

Since C_{11}' has changed sign, being now -0.79 as compared with $+1.00$ in the original equations, the signs of the Δk's in e_1' should be reversed, giving the following combination of signs to be tried next:

$$\begin{vmatrix} - & - & - \\ + & + & + \\ + & + & + \end{vmatrix}$$

The new revised equations are:

$$1.8x'' + 2.7y'' + 2.7z'' = \ \ 12.85$$

$$1.1x'' - 4.5y'' + 2.2z'' = -13.10$$

$$1.1x'' + 2.2y'' - 0.9z'' = \ \ 8.05$$

whence

$$D'' = \quad 1.8\ (-0.79) + 1.1\ (\ \ 8.37) + 1.1\ (\ \ 18.09) = 27.684$$

$$x'' = \frac{1}{27.684}[12.85(-0.79) - 13.10(\ \ 8.37) + 8.05(\ \ 18.09)] = \ 0.933$$

$$y'' = \frac{1}{27.684}[12.85(\ \ 3.41) - 13.10(-4.59) + 8.05(-0.99)] = \ 3.467$$

$$z'' = \frac{1}{27.684}[12.85(\ \ 7.37) - 13.10(-0.99) + 8.05(-11.07)] = \ 0.670$$

$$\textit{Test:} \quad \delta x'' = \frac{1}{-27.684}(-0.79e_1'' + 8.37e_2'' + 18.09e_3'')$$

where $$e_i'' = 0.933\Delta k_{i1} + 3.467\Delta k_{i2} + 0.670\Delta k_{i3}$$

It is now seen that every term $\dfrac{\partial x}{\partial k_i}\Delta k_i$ in the expression for $\delta x''$ is negative for the signs used, namely,

$$\begin{vmatrix} - & - & - \\ + & + & + \\ + & + & + \end{vmatrix}$$

and therefore the true maximum error (negative) is $\Delta x^\circ = 0.933 - 2.000 = -1.067$.

Example No. 10. Find the true maximum positive error in y°, using the data of Example No. 5.

Solution: In Section 12 the expressions for the primary errors were found to be as follows:

$$\delta y = \frac{1}{-J}\begin{vmatrix} \left.\dfrac{\partial F_1}{\partial x}\right|_o & E_1 \\ \left.\dfrac{\partial F_2}{\partial x}\right|_o & E_2 \end{vmatrix}$$

and

$$\delta x = \frac{1}{-J}\begin{vmatrix} E_1 & \left.\dfrac{\partial F_1}{\partial y}\right|_o \\ E_2 & \left.\dfrac{\partial F_2}{\partial y}\right|_o \end{vmatrix}$$

where

$$J = \begin{vmatrix} \left.\dfrac{\partial F_1}{\partial x}\right|_o & \left.\dfrac{\partial F_1}{\partial y}\right|_o \\ \left.\dfrac{\partial F_2}{\partial x}\right|_o & \left.\dfrac{\partial F_2}{\partial y}\right|_o \end{vmatrix}$$

$$\left.\frac{\partial F_1}{\partial x}\right|_o = 2k_1^\circ x^\circ + k_2^\circ = 12$$

$$\left.\frac{\partial F_2}{\partial x}\right|_o = 2(k_4^\circ)^2 x^\circ + k_4^\circ k_5^\circ y^\circ = 28.5$$

$$\left.\frac{\partial F_1}{\partial y}\right|_o = k_3^o = -3$$

$$\left.\frac{\partial F_2}{\partial y}\right|_o = 2(k_5^o)^2 y^o + k_4^o k_5^o x^o = 31$$

$$E_1 = (x^o)^2 \Delta k_1 + x^o \Delta k_2 + y^o \Delta k_3 = 4.1$$

and $E_2 = [2k_4^o(x^o)^2 + k_5^o x^o y^o]\,\Delta k_4 + [2k_5^o(y^o)^2 + k_4^o x^o y^o]\,\Delta k_5 = -20.45$. The initial values of the unknowns were $x^o = 5$, $y^o = 2$; and the calculated primary errors were $\delta x = -0.144$ and δy (max.) $= +0.792$.

The apparent maximum error will be calculated by using the same signs with the Δk's as were required in obtaining the maximum positive primary error in y^o, the modified values of the arbitrary constants being as follows:

$$k_1' = 1 + .1 = 1.1$$

$$k_2' = 2 + .2 = 2.2$$

$$k_3' = -3 + .3 = -2.7$$

$$k_4' = 1.5 - .15 = 1.35$$

$$k_5' = 2 - .2 = 1.8$$

In accordance with Equations (25) and (26), Section 12, the revised equations are

$$1.1(x')^2 + 2.2x' - 2.7y' = 29 \qquad (40)$$

$$1.8225(x')^2 + 2.43x'y' + 3.24(y')^2 = 102.25 \qquad (41)$$

In order to obtain the solution of these equations, the second method of Section 16 will be used. Approximate values of the unknowns, as found from the primary errors are 4.856 (for x') and 2.792 (for y'). Using these values in the left-hand side of Equations (40) and (41), the following equations are obtained:

$$1.1x^2 + 2.2x - 2.7y = 29.0836$$

$$1.8225x^2 + 2.43xy + 3.24y^2 = 101.1782$$

By varying the right-hand terms of these equations to make them agree with Equations (40) and (41), and calculating the primary errors in the assumed values of the unknowns, the solution of the revised equations may be closely approximated. The required variations in the right-hand terms are: $\Delta\nu_1 = -0.0836$ and $\Delta\nu_2 = 1.0718$.

The values of the corresponding primary errors, or corrections, $\delta'x$ and $\delta'y$, in accordance with Equation (31b), Section 13, are then as follows:

$$\delta'x = \frac{1}{J'}\begin{vmatrix} \Delta\nu_1 & \dfrac{\partial F_1}{\partial y} \\ \Delta\nu_2 & \dfrac{\partial F_2}{\partial y} \end{vmatrix}, \text{ and } \delta'y = \frac{1}{J'}\begin{vmatrix} \dfrac{\partial F_1}{\partial x} & \Delta\nu_1 \\ \dfrac{\partial F_2}{\partial x} & \Delta\nu_2 \end{vmatrix}$$

where
$$\frac{\partial F_1}{\partial x} = 2k_1'x + k_2' = 12.8832$$

$$\frac{\partial F_2}{\partial x} = 2(k_4')^2x + k_4'k_5'y = 24.48468$$

$$\frac{\partial F_1}{\partial y} = k_3' = -2.7$$

and
$$\frac{\partial F_2}{\partial y} = 2(k_5')^2y + k_4'k_5'x = 29.89224$$

The value of the Jacobian is therefore:

$$J' = \begin{vmatrix} 12.8832 & -2.7 \\ 24.48468 & 29.89224 \end{vmatrix} = 451.2163$$

whence
$$\delta'x = \frac{1}{451.2163}\begin{vmatrix} -0.0836 & -2.7 \\ 1.0718 & 29.89224 \end{vmatrix} = +0.0009$$

and
$$\delta'y = \frac{1}{451.2163}\begin{vmatrix} 12.8832 & -0.0836 \\ 24.48468 & 1.0718 \end{vmatrix} = +0.0351$$

The corrected values of x' and y' are thus found to be 4.8569 and 2.8271, respectively, and when used in Equations (40) and (41) yield the following results:*

$$1.1(x')^2 + 2.2x' - 2.7y' = 29.00035$$

$$1.8225(x')^2 + 2.43x'y' + 3.24(y')^2 = 102.25354$$

The apparent maximum error in y^o is therefore $2.8271 - 2.00 = +0.8271$.

In order to determine whether or not this is the true maximum error it is necessary to investigate the signs of the partial derivatives

*If greater precision in the solution of the revised equations is desired, a final set of corrections may be obtained by again varying the right-hand terms, putting $\Delta\nu_1 = -0.00035$ and $\Delta\nu_2 = -0.00354$.

$\frac{\partial y}{\partial k_1} \ldots \frac{\partial y}{\partial k_5}$, using the modified values of the arbitrary constants and of the unknowns. The modified value of the Jacobian is as follows:

$$J' = \begin{vmatrix} 2.2x' + 2.2 & -2.7 \\ 3.645x' + 2.43y' & 6.48y' + 2.43x' \end{vmatrix} = \begin{vmatrix} 12.88 & -2.7 \\ 24.57 & 30.12 \end{vmatrix}$$

It is now seen that no changes have occurred in the signs of the minors of the Jacobian. Furthermore, since none of the unknowns have changed sign, it is evident that all of the error-coefficients have retained their original sign. A further increase in Δy can only be obtained by increasing the Δk's beyond their assumed maximum values, and hence the true maximum error in y° is $+0.8271$.

V. Errors in Functions of the Unknowns

18. *Errors in Functions of the Unknowns.*—The methods which have been used in the preceding sections, in connection with the calculation of errors in the unknowns, are applicable, with slight modifications, to the determination of errors in functions containing one or more of the unknowns $x_1 \ldots x_n$.

Let $\theta = h(x_1, x_2, \ldots; k_1, k_2, \ldots)$ be such a function; and let $\theta^\circ = h(x_1^\circ, x_2^\circ, \ldots; k_1^\circ, k_2^\circ, \ldots)$. The primary error in θ° is then given by

$$\delta\theta = \left(\frac{\partial\theta}{\partial x_1}\delta x_1 + \frac{\partial\theta}{\partial x_2}\delta x_2 + \ldots + \frac{\partial\theta}{\partial k_1}\Delta k_1 + \frac{\partial\theta}{\partial k_2}\Delta k_2 + \ldots \right)\Bigg|_\circ \quad (42)$$

In order to determine the maximum primary error in θ°, the initial values of the unknowns $x_1^\circ, x_2^\circ, \ldots$ (found by solving the original equations), and the expressions for $\delta x_1, \delta x_2, \ldots$ are first obtained. These results are then substituted in Equation (42), and the terms are re-arranged so as to show each Δk with its proper coefficient. In this way, the expression for $\delta\theta$ is reduced to a linear function of the increments $\Delta k_1 \ldots \Delta k_p$, from which one can determine by inspection the necessary signs which the Δk's must have in order that the primary error in the given function shall be a maximum.

The apparent maximum error may be computed by solving for $x_1' \ldots x_n'$ in the revised equations by one of the methods of Section 16, and then finding the corresponding value of θ', whence $\Delta\theta^\circ = \theta' - \theta^\circ$.

If the true maximum error in θ° is required, a criterion expression is found from the expression for $\delta\theta$ by replacing the initial values of

the arbitrary constants and of the unknowns by the corresponding modified values. The signs of the error-coefficients in the criterion expression, i. e., of $\frac{\partial\theta}{\partial k_1} \ldots \frac{\partial\theta}{\partial k_p}$, are then examined, and the method described in Section 17 is followed in determining the signs and magnitudes which the Δk's must have in order to produce the true maximum error.

The probable error in $\theta°$, following the method described in Section 15, is obtainable from the terms in the expression for $\delta\theta$, after the latter has been reduced to the proper form for determining the maximum primary error. The probable error, $\lambda\theta°$, is then found by taking the square root of the sum of the squares of the different terms in this expression.

In order to illustrate the calculation of the maximum and probable errors in functions containing the unknowns of a system of simultaneous equations, the following example, dealing with the bending moments in a continuous frame, will be solved.

Example No. 11. Find the maximum error in M_{yz}, Fig. 1, due to possible variations of 10 per cent in the EI/L values of the different members; also find the probable error in M_{yz} due to probable errors of ± 5 per cent in the EI/L values.

The maximum moments will be computed for a uniform live load, the effects of dead load and side-sway being neglected.

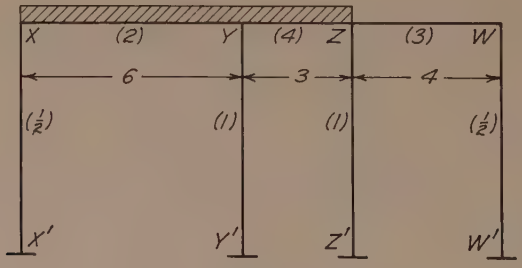

Fig. 1

The relative span lengths and EI/L values are shown in the figure. Let M_2 denote the fixed-end moment in the middle span, and let $k = EI/L$. Also denote the rotations of joints X, Y, Z and W by

$\frac{1}{2}x$, $\frac{1}{2}y$, $\frac{1}{2}z$, and $\frac{1}{2}w$, respectively. The slope-deflection equations of equilibrium will then be as follows:*

$$2(k_{xx'} + k_{xy})x + k_{xy}y = 4M_2 \quad (43)$$

$$k_{xy}x + 2(k_{xy} + k_{yy'} + k_{ys})y + k_{yz}z = -3M_2 \quad (44)$$

$$k_{yz}y + 2(k_{yz} + k_{zz'} + k_{zw})z + k_{zw}w = -M_2 \quad (45)$$

$$k_{zw}z + 2(k_{zw} + k_{ww'})w = 0 \quad (46)$$

When the theoretical k-values shown in Fig. 1 are substituted, these equations become

$$5x^\circ + 2y^\circ = 4M_2 \quad (47)$$

$$2x^\circ + 14y^\circ + 4z^\circ = -3M_2 \quad (48)$$

$$4y^\circ + 16z^\circ + 3w^\circ = -M_2 \quad (49)$$

$$3z^\circ + 7w^\circ = 0 \quad (50)$$

for which the following solution is obtained:

$$D = \begin{vmatrix} 5 & 2 & 0 & 0 \\ 2 & 14 & 4 & 0 \\ 0 & 4 & 16 & 3 \\ 0 & 0 & 3 & 7 \end{vmatrix} = 6238,$$

$$x^\circ = \frac{M_2}{D}\begin{vmatrix} 4 & 2 & 0 & 0 \\ -3 & 14 & 4 & 0 \\ -1 & 4 & 16 & 3 \\ 0 & 0 & 3 & 7 \end{vmatrix} = \frac{M_2}{D}[4(1330) - 3(-206) - 1(56) + 0(-24)] = 0.9429M_2$$

$$y^\circ = \frac{M_2}{D}\begin{vmatrix} 5 & 4 & 0 & 0 \\ 2 & -3 & 4 & 0 \\ 0 & -1 & 16 & 3 \\ 0 & 0 & 3 & 7 \end{vmatrix} = \frac{M_2}{D}[4(-206) - 3(515) - 1(-140) + 0(60)] = -0.3573M_2$$

*See Bulletin No. 108, Engineering Experiment Station, Univ. of Ill., "Analysis of Statically Indeterminate Structures by the Slope Deflection Method," by W. M. Wilson, F. E. Richart, and Camillo Weiss.

$$z^\circ = \frac{M_2}{D}\begin{vmatrix} 5 & 2 & 4 & 0 \\ 2 & 14 & -3 & 0 \\ 0 & 4 & -1 & 3 \\ 0 & 0 & 0 & 7 \end{vmatrix} = \frac{M_2}{D}[4(56)-3(-140) -1(462)+0(-198)] = 0.0292M_2$$

$$w^\circ = \frac{M_2}{D}\begin{vmatrix} 5 & 2 & 0 & 4 \\ 2 & 14 & 4 & -3 \\ 0 & 4 & 16 & -1 \\ 0 & 0 & 3 & 0 \end{vmatrix} = \frac{M_2}{D}[4(-24)-3(60) -1(-198)+0(976)] = -0.0125M_2$$

The function being studied is M_{yz}, as defined by the equation

$$M_{yz} = k_{yz}(2y + z) - M_2 \tag{51}$$

The initial value of this quantity is as follows:

$$M^\circ_{yz}\,(\text{max.}) = 4(-0.6854M_2) - M_2 = -3.742M_2$$

The expressions for the primary errors in the unknowns, as given by Equation (11b), are as follows:

$$\delta x = \frac{1}{-D}(\ 1330e_1 - 206e_2 + \ 56e_3 - \ 24e_4) \tag{52}$$

$$\delta y = \frac{1}{-D}(-206e_1 + 515e_2 - 140e_3 + \ 60e_4) \tag{53}$$

$$\delta z = \frac{1}{-D}(\ \ 56e_1 - 140e_2 + 462e_3 - 198e_4) \tag{54}$$

$$\delta w = \frac{1}{-D}(-\ 24e_1 + \ 60e_2 - 198e_3 + 976e_4) \tag{55}$$

From Equations (43) to (46), inclusive, the following values for $e_1 \ldots . e_4$ are found:

$$e_1 = 2x^\circ\Delta k_{xx'} + (2x^\circ + y^\circ)\Delta k_{xy}$$

$$e_2 = (x^\circ + 2y^\circ)\Delta k_{xy} + 2y^\circ\Delta k_{yy'} + (2y^\circ + z^\circ)\Delta k_{yz}$$

$$e_3 = (y^\circ + 2z^\circ)\Delta k_{yz} + 2z^\circ\Delta k_{zz'} + (2z^\circ + w^\circ)\Delta k_{zw}$$

$$e_4 = (z^\circ + 2w^\circ)\Delta k_{zw} + 2w^\circ\Delta k_{ww'}$$

Upon substituting the computed values of x°, y°, z°, and w°, these quantities become:

$$e_1 = (1.886\Delta k_{xx'} + 1.528\Delta k_{xy})M_2 \tag{56}$$

$$e_2 = (-0.715\Delta k_{yy'} + 0.228\Delta k_{xy} - 0.685\Delta k_{yz})M_2 \tag{57}$$

$$e_3 = (0.0584\Delta k_{zz'} - 0.299\Delta k_{yz} + 0.046\Delta k_{zw})M_2 \tag{58}$$

$$e_4 = (-0.025\Delta k_{ww'} + 0.0042\Delta k_{zw})M_2 \tag{59}$$

From Equations (42) and (51), the primary error in M°_{yz} is

$$\delta M_{yz} = k^\circ_{yz}(2\delta y + \delta z) + (2y^\circ + z^\circ)\Delta k_{yz}$$

or, since $k^\circ_{yz} = 4$, and $2y^\circ + z^\circ = -0.6854M_2$

$$\delta M_{yz} = 8\delta y + 4\delta z - 0.6854\, M_2\, \Delta k_{yz} \tag{60}$$

The value of the first two terms in Equation (60), as given by Equations (53) and (54), is as follows:

$$8\delta y + 4\delta z = 0.2285\, e_1 - 0.571\, e_2 - 0.1167\, e_3 + 0.0501\, e_4$$

and by using the values given in Equations (56) to (59), inclusive, and substituting in Equation (60), one obtains the following reduced expression for the primary error in M°_{yz} in terms of the various Δk's:

$$\begin{aligned}\delta M_{yz} = M_2(&0.431\Delta k_{xx'} + 0.408\Delta k_{yy'} - 0.0068\Delta k_{zz'} - 0.0012\Delta k_{ww'} \\ &+ 0.220\Delta k_{xy} - 0.258\Delta k_{yz} - 0.0052\Delta k_{zw})\end{aligned} \tag{61}$$

From this expression it is seen that the specified variations in the k-values must have the following signs in order to produce the maximum primary error in M°_{yz}:

$$\Delta k_{xx'} = -0.05, \qquad \Delta k_{ww'} = +0.05, \qquad \Delta k_{yz} = +0.4$$

$$\Delta k_{yy'} = -0.1, \qquad \Delta k_{xy} = -0.2, \qquad \Delta k_{zw} = +0.3$$

$$\Delta k_{zz'} = +0.1,$$

Upon substituting the foregoing values into Equation (61), the maximum primary error in M°_{zy} is found to be $-0.212M_2$, which is equivalent to an error of 5.7 per cent.

In order to determine the apparent maximum error, the revised equations are written and solved, namely,

$$\begin{aligned} 4.5x' + 1.8y' &= 4M_2 \\ 1.8x' + 14.2y' + 4.4z' &= -3M_2 \\ 4.4y' + 17.6z' + 3.3w' &= -M_2 \\ 3.3z' + 7.7w' &= 0 \end{aligned}$$

The solution is as follows:

$$D' = 4.5(1620.674) + 1.8(-224.334) = 6889.2318$$

$$x' = \frac{M_2}{D'}[4(1620.674) - 3(-224.334) - 1(60.984)] = 1.0298M_2$$

$$y' = \frac{M_2}{D'}[4(-224.334) - 3(560.835) - 1(-152.46)] = -0.3523M_2$$

$$z' = \frac{M_2}{D'}[4(60.984) - 3(-152.46) - 1(467.082)] = 0.0340M_2$$

$$w' = \frac{M_2}{D'}[4(-26.136) - 3(65.35) - 1(-200.178)] = -0.0146M_2$$

and
$$M'_{yz} = k'_{yz}(2y' + z') - M_2$$
$$= 4.4(-0.6707M_2) - M_2 = -3.951M_2$$

The apparent maximum error is therefore

$$\Delta M^{\circ}_{yz} = -3.951M_2 - (-3.742M_2) = -0.209M_2$$

Test: $\delta M'_{yz} = k'_{yz}(2\delta y' + \delta z') + (2y' + z')\Delta k_{yz}$
$$= 8.8\delta y' + 4.4\delta z' - 0.6707M_2\Delta k_{yz}$$

$$\delta y' = \frac{1}{-D'}(-224.334e'_1 + 560.835e'_2 - 152.46e'_3 + 65.34e'_4)$$

$$\delta z' = \frac{1}{-D'}(60.984e'_1 - 152.46e'_2 + 467.082e'_3 - 200.178e'_4)$$

where
$$e'_1 = (2.060\Delta k_{xx'} + 1.708\Delta k_{xy})M_2$$
$$e'_2 = (-0.702\Delta k_{yy'} + 0.328\Delta k_{xy} - 0.668\Delta k_{yz})M_2$$
$$e'_3 = (0.068\Delta k_{zz'} - 0.284\Delta k_{yz} + 0.053\Delta k_{zw})M_2$$
$$e'_4 = (-0.029\Delta k_{ww'} + 0.005\Delta k_{zw})M_2$$

Also, $8.8\delta y' + 4.4\delta z' = 0.2475e'_1 - 0.619e'_2 - 0.1034e'_3 + 0.0476e'_4$ whence the reduced expression for $\delta M'_{yz}$ is as follows:

$$\delta M'_{yz} = (0.510\Delta k_{xx'} + 0.444\Delta k_{yy'} - 0.0070\Delta k_{zz'} - 0.0014\Delta k_{ww'} + 0.220\Delta k_{xy} - 0.229\Delta k_{yz} - 0.0053\Delta k_{zw})M_2$$

Since the separate terms in the foregoing expression will all be negative when the Δk's are given the signs which were used in com-

puting the apparent maximum error, the latter quantity is identical with the true maximum error.

Probable Error in M°_{yz}. The probable error in M°_{yz}, found by substituting the specified probable errors of 5 per cent for the Δk's in the expression for the primary error, Equation (61), and taking the square root of the sum of the squares of the resulting terms, is as follows:

$$M^{\circ}_{yz} = \pm M_2 \sqrt{\begin{aligned}&[0.431(0.025)]^2 + [0.408(0.05)]^2 + [0.0068(0.05)]^2 \\ &+ [0.0012(0.025)]^2 + [0.220(0.1)]^2 \\ &+ [(0.258(0.2)]^2 + [0.0052(0.15)]^2 \\ &= \pm 0.065 M_2\end{aligned}}$$

VI. Conclusions

19. *Comparison of Results.*—For purposes of comparison, the results of the preceding examples are tabulated below.

Example No.	Quantity Investigated	Maximum Primary Error*	Apparent Maximum Error*	True Maximum Error*	Probable Error†
4, 7 & 9	$x^{\circ} = 2.00$	$+1.050$ -1.050	$+1.129$ -1.000	$+1.129$ -1.067	± 0.298
5 & 10	$y^{\circ} = 2.00$	$+0.792$	$+0.827$	$+0.827$	± 0.221
11	$M^{\circ}_{yz} = -3.742M_2$	$-0.212M_2$	$-0.209M_2$	$-0.209M_2$	$\pm 0.065M_2$

*Assuming maximum errors of 10 per cent in the arbitrary constants.
†Assuming probable errors of ±5 per cent in the arbitrary constants.

20. *Conclusions.*—From the results of the foregoing investigation the following conclusions are offered:

(1) The calculation of the maximum primary error and the probable error in a quantity governed by simultaneous equations is sufficiently simple to lend practical value to their determination.

(2) When the errors in the arbitrary constants are in the neighborhood of 10 per cent, the maximum primary error will probably not differ from the true maximum error by appreciably more than 10 per cent.

(3) The apparent maximum error, although requiring more time for its determination than the maximum primary error, is a more satisfactory approximation to the true maximum error over a wide

range of conditions, and in a majority of instances is identical with the true maximum error. It is worthy of note that the theoretical effect of a Δk whose coefficient undergoes a change in sign while x_m is changing from x_m° to x_m' (or while a given function θ is changing from θ° to θ') may be considerable; however, the apparent maximum error is probably a sufficiently good approximation to the true maximum error in most practical cases, in view of the unavoidable uncertainty in estimating the magnitudes of the errors in the arbitrary constants.

These conclusions are based upon a study of relatively strong systems of equations, and therefore should not be applied unreservedly to systems approaching the condition of linear dependence.

Appendix A

1. *Errors of Higher Orders in the Unknowns of Linear Equations.*—The second-order total differentials of the functions represented by Equations (5) to (5n), Section 7, are as follows:

$$d^2f_1 = a_{11}d^2x_1 + \ldots + a_{1m}d^2x_m + \ldots + a_{1n}d^2x_n + 2dx_1da_{11} + \ldots + 2dx_mda_{1m} + \ldots + 2dx_nda_{1n} = 0 \quad (62)$$

. . . . . .

$$d^2f_m = a_{m1}d^2x_1 + \ldots + a_{mm}d^2x_m + \ldots + a_{mn}d^2x_n + 2dx_1da_{m1} + \ldots + 2dx_mda_{mm} + \ldots + 2dx_nda_{mn} = 0 \quad (62m)$$

. . . . . .

$$d^2f_n = a_{n1}d^2x_1 + \ldots + a_{nm}d^2x_m + \ldots + a_{nn}d^2x_n + 2dx_1da_{n1} + \ldots + 2dx_mda_{nm} + \ldots + 2dx_nda_{nn} = 0 \quad (62n)$$

By means of the relations $\left.\frac{d^2x_i}{2!}\right|_o = \delta^2x_i$, $dx_i\Big|_o = \delta x_i$, and $da_{ij} = \Delta a_{ij}$, the following equations are then obtained:

$$a^o_{11}\delta^2x_1 + \ldots + a^o_{1m}\delta^2x_m + \ldots + a^o_{1n}\delta^2x_n + \sum_{j=1}^{n}\delta x_j\Delta a_{1j} = 0 \quad (63)$$

. . . . . .

$$a^o_{m1}\delta^2x_1 + \ldots + a^o_{mm}\delta^2x_m + \ldots + a^o_{mn}\delta^2x_n + \sum_{j=1}^{n}\delta x_j\Delta a_{mj} = 0 \quad (63m)$$

. . . . . .

$$a^o_{n1}\delta^2x_1 + \ldots + a^o_{nm}\delta^2x_m + \ldots + a^o_{nn}\delta^2x_n + \sum_{j=1}^{n}\delta x_j\Delta a_{nj} = 0 \quad (63n)$$

It will be noted that the determinant of the coefficients of the δ^2x's in Equations (63) to (63n), inclusive, is identical with that of Equations (6) to (6n), in Section 7, and the terms preceded by the summation signs differ from the corresponding terms in Equations (6) to (6n), inclusive, only in having the unknowns $x_1 \ldots x_n$, in the latter, replaced by the primary errors, $\delta x_1 \ldots \delta x_n$, respectively.

By letting $\delta e_i = \sum_{j=1}^{n} \delta x_j \Delta a_{ij}$ (see Equation 7), the secondary error in x_m°, as found from Equations (63) to (63n), is then as follows:

$$\delta^2 x_m = \frac{1}{-D} \begin{vmatrix} a_{11}^\circ & \cdots & \delta e_1 & \cdots & a_{1n}^\circ \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ a_{m1}^\circ & \cdots & \delta e_m & \cdots & a_{mn}^\circ \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ a_{n1}^\circ & \cdots & \delta e_n & \cdots & a_{nn}^\circ \end{vmatrix} \tag{64a}$$

or, using the co-factor notation,

$$\delta^2 x_m = \frac{1}{-D} \sum_{i=1}^{n} C_{im} \delta e_i \tag{64b}$$

In a similar way it can readily be shown that the errors $\delta^3 x_m \ldots$. $\delta^N x_m$ are given by

$$\delta^3 x_m = \frac{1}{-D} \sum_{i=1}^{n} C_{im} \delta^2 e_i \tag{65}$$

$$\delta^N x_m = \frac{1}{-D} \sum_{i=1}^{n} C_{im} \delta^{N-1} e_i \tag{66}$$

where $$\delta^2 e_i = \sum_{j=1}^{n} \delta^2 x_j \Delta a_{ij} \tag{67}$$

and $$\delta^{N-1} e_i = \sum_{j=1}^{n} \delta^{N-1} x_j \Delta a_{ij} \tag{68}$$

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